The Sum-the-Odds Theorem with Application to a Stopping Game of Sakaguchi

Thomas S. Ferguson
University of California at Los Angeles

Abstract: The optimal stopping problem of maximizing the probability of stopping on the last success of a finite sequence of independent Bernoulli trials has been studied by Hill and Krengel (1992), Hsiau and Yang (2000) and Bruss (2000). The optimal stopping rule of Bruss stops when the sum of the odds of future successes is less than one. This Sum-the-Odds Theorem is extended in several ways. First, an infinite number of Bernoulli trials is allowed. Second, the payoff for not stopping is allowed to be different from the payoff of stopping on a success that is not the last success. Third, the Bernoulli variables are allowed to be dependent. Fourth, the model is generalized to allow at each stage other dependent random variables to be observed that may influence the assessment of the probability of success at future stages. Finally, application is made to a game of Sakaguchi (1984) in which two players vie for predicting the last success, but in which one of the players is given priority of acting first.

The author dedicates this paper to the memory of Professor Minoru Sakaguchi (1926-2009) with great respect for his many contributions to the fields of optimal stopping and game theory that have inspired generations of students and scholars.

1. Description of the Problem. Fix a positive integer n, and let $X_1, X_2, \ldots, X_n$ be Bernoulli random variables. The X_i are observed sequentially. The problem is to find a stopping rule N to maximize the probability of stopping at the last success. In Hill and Krengel (1992), Hsiau and Yang (2000) and Bruss (2000), the X_i are taken to be independent. Here we investigate the problem for dependent X_i.

Suppose that the X_i are independent, and let $p_i = \mathrm{P}(X_i = 1)$. The optimal stopping rule of Bruss (2000) is simply

$$N^* = \min\{k \geq 1 : X_k = 1 \text{ and } \sum_{i=k+1}^{n} \frac{p_i}{1-p_i} \leq 1\}. \tag{1}$$

(If $p_i = 1$, $p_i/(1-p_i)$ is taken to be $+\infty$.) In other words, stop at the first success ($X_k = 1$) for which the sum of the odds of success for future X_i is less than or equal to 1. If the

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sum of the odds is equal to 1, we are actually indifferent between stopping and continuing. This result is referred to as the Sum-the-Odds Theorem. The Classical Secretary Problem occurs if $p_i = 1/i$ for $i = 1, \ldots, n$, and in this case it is known that $k/n \to e^{-1}$, where the optimal rule selects the first success after stage k. More remarkable, in Hill and Krengel (1992) and in Bruss (2003), it is seen that the optimal probability of stopping on the last success is at least e^{-1} whatever be the values of the p_i, provided that the sum of the odds is at least 1 (Bruss) or that the probability of at least one success is 1 (Hill and Krengel). See also Bruss and Paindaveine (2000). The paper of Tamaki, Wang and Kurushima (2008) contains an extension to allow a random horizon, whereas Bruss and Louchard (2009) propose a model for unknown odds and study the impact of sequential estimation in the corresponding odds algorithm. Newer developments surrounding the Sum-the-Odds Theorem, including the notion of multiplicative odds studied by Tamaki (2010) as well as multiple selection chances (Ano et al. (2010)) are summarized in Dendievel (2013). A related problem of stopping in continuous time, called the last arrival problem, has a fascinating solution presented by Bruss and Yor (2012).

There are two small improvements one may make on the original Sum-the-Odds Theorem. First, we may set the payoff for not stopping to be different from the payoff of stopping on a success that is not the last success. For the classical secretary problem, this generalization is due to Sakaguchi (1984). Let ω represent the payoff for not stopping. If $\omega \geq 1$, it is clear that it is optimal never to stop. So for simplicity, we take $\omega < 1$, though ω may be allowed to be negative provided it is forbidden to stop on a failure.

Secondly, we may allow an infinite number of Bernoulli variables. The same rule (1) is optimal if n is replaced by ∞. However, with an infinite number of X_i, it may happen that there is no last $X_i = 1$. By the Borel-Cantelli Lemma, this happens with independent X_i if and only if $\sum_1^\infty p_i = \infty$. In this case the sum of the odds is infinite so that the rule N^* never stops and the payoff is assumed in that case to be ω. If $\sum_1^\infty p_i < \infty$, the rule N^* eventually stops, and is optimal.

With both these features added to the problem, the optimal stopping rule in the case $\sum_1^\infty p_i < \infty$ becomes

$$N^* = \min\{k \geq 1 : X_k = 1 \text{ and } \sum_{i=k+1}^{\infty} \frac{p_i}{1-p_i} \leq 1 - \omega\}. \tag{2}$$

This result follows from the main theorem below. If $\sum_1^\infty p_i = \infty$, the optimal rule is trivial, namely: if $\omega \geq 0$, one never stops, and the payoff is ω; if $\omega < 0$, one might as well stop at the first success, and the payoff is zero.

The objective of this paper is to extend this result to the case where the X_i are dependent. In addition, we generalize the problem to allow more information to be given to the decision maker at each stage. At stage i, in addition to observing X_i, the decision maker observes other random variables that may influence his assessment of the probability of success at future stages.

Hsiau and Yang (2002) explore the problem of maximizing the probability of stopping on the last success when the observations form a Markov chain of Bernoulli random variables. The problem is solved completely when the chain is homogeneous and in some nonhomogeneous cases as well. The method of derivation differs from the method given here, and some of their results are not obtainable from the general theorem given here.

2. A General Model for Best Choice Problems. The method we use to solve the problem is as follows. First we modify the problem, as was done by Dynkin (1963) in his treatment of the secretary problem, by not allowing stopping on a failure. This seemingly innocuous modification changes the secretary problem into a monotone stopping problem. Then we may apply a simple result that gives conditions for the one-stage look-ahead rule to be optimal in a monotone problem. See for example Chow, Robbins and Siegmund (1971) or Ferguson (2006).

When stopping on failures is forbidden, we must change the notion of a "stage". A stage is defined to contain all the observations up to and including the next success, if any. We model this as follows. For $i = 1, 2, \ldots$, let Z_i denote the set of random variables observed after success $i-1$ up to and including success i. If there are less than i successes, we let $Z_i = 0$, where 0 is a special absorbing state. Thus we treat the following general model.

Let $Z_1, Z_2, \ldots$ be a stochastic process on an arbitrary space with an absorbing state called 0. We make the assumption that with probability one the process will eventually be absorbed at 0. We observe the process sequentially and we wish to predict one stage in advance when the state 0 will first be hit. If we predict correctly, we win 1, if we predict incorrectly we win nothing, and if the process hits 0 before we predict, we win ω, where $\omega < 1$. This is a stopping rule problem in which stopping at stage n yields the payoff

$$\begin{aligned} &Y_n = \omega \mathrm{I}(Z_n = 0) + \mathrm{I}(Z_n \neq 0)\mathrm{P}(Z_{n+1} = 0|\mathcal{G}_n) \quad \text{for } n = 1, 2, \ldots \\ &Y_\infty = \omega \end{aligned} \tag{3}$$

where $\mathrm{I}(A)$ represents the indicator function of the event A and $\mathcal{G}_n = \sigma(Z_1, \ldots, Z_n)$, the σ-field generated by $Z_1, \ldots, Z_n$. The assignment $Y_\infty = \omega$ means that if we never stop, we win ω. To find the one-stage look-ahead rule (the 1-sla), we evaluate

$$\mathrm{E}(Y_{n+1}|\mathcal{G}_n) = \omega \mathrm{P}(Z_{n+1} = 0|\mathcal{G}_n) + \mathrm{P}(Z_{n+1} \neq 0, Z_{n+2} = 0|\mathcal{G}_n) \tag{4}$$

The 1-sla calls for stopping at stage n if $Y_n \geq \mathrm{E}(Y_{n+1}|\mathcal{G}_n)$. On the set $\{Z_n = 0\}$, this reduces to $\omega \geq \omega$ which is always true. On the set $\{Z_n \neq 0\}$, this reduces to

$$(1 - \omega)\mathrm{P}(Z_{n+1} = 0|\mathcal{G}_n) \geq \mathrm{P}(Z_{n+1} \neq 0, Z_{n+2} = 0|\mathcal{G}_n) \tag{5}$$

Therefore the 1-sla is

$$N_1 = \min\{n : Z_n = 0 \text{ or } (Z_n \neq 0 \text{ and } \frac{\mathrm{P}(Z_{n+1} \neq 0, Z_{n+2} = 0|\mathcal{G}_n)}{\mathrm{P}(Z_{n+1} = 0|\mathcal{G}_n)} \leq 1 - \omega)\} \tag{6}$$

If $\mathrm{P}(Z_{n+1} = 0|\mathcal{G}_n) = 0$ on $\{Z_n \neq 0\}$, then it is a mistake to stop at n since we can do at least as well by continuing one more step. Therefore in this and subsequent formulae, we take the ratio in (6) to be $+\infty$ when $\mathrm{P}(Z_{n+1} = 0|\mathcal{G}_n) = 0$, even if the numerator is zero as well.

The problem is said to be *monotone* if, when the 1-sla calls for stopping at any stage, then it will continue to call for stopping at all future stages no matter what the future observations turn out to be. Specifically, the problem is monotone if

$$A_1 \subset A_2 \subset A_3 \subset \cdots \quad \text{a.s.} \tag{7}$$

where for all n, $A_n = \{Y_n \geq \mathrm{E}(Y_{n+1}|\mathcal{G}_n)\}$. From this we see that a sufficient condition for the problem to be monotone is

$$\frac{\mathrm{P}(Z_{n+1} \neq 0, Z_{n+2} = 0|\mathcal{G}_n)}{\mathrm{P}(Z_{n+1} = 0|\mathcal{G}_n)} \qquad \text{is nonincreasing in } n \text{ a.s.} \tag{8}$$

One of the basic theorems in the theory of optimal stopping gives conditions under which the 1-sla is optimal for monotone stopping rule problems. In the present context, we may use the result that for a monotone stopping problem with observations $Z_1, Z_2, \ldots$ and payoff functions, $Y_1, Y_2, \ldots, Y_\infty$, the 1-sla is optimal if $\sup_n |Y_n|$ has finite expectation and $\lim_{n\to\infty} Y_n = Y_\infty$ a.s. (See the electronic text of Ferguson (2006), Chapter 5, Theorem 2 and its Corollary.) In the problem we are considering, $|Y_n|$ is bounded by $1 + |\omega|$, and $\lim_{n\to\infty} Y_n = \omega = Y_\infty$ a.s., since we are assuming that the process is absorbed at zero with probability one. Thus,

Theorem 1. *Suppose that the process $Z_1, Z_2, \ldots$ has an absorbing state, 0, such that $\mathrm{P}(Z_n$ is absorbed at $0) = 1$. Suppose the stopping problem with reward sequence (3) satisfies (8). Then the 1-sla (6) is optimal.*

3. Application to the Sum-the-Odds Theorem. We return to the original problem of stopping on the last success of a sequence of possibly dependent Bernoulli trials, $X_1, X_2, \ldots$. We model the information given to the decision maker through an increasing sequence of σ-fields, $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$, and allow him to use a stopping rule adapted to this sequence. We assume that for every j the event $\{X_j = 1\}$ is in $\mathcal{F}_j$. In the formulation of Section 2, if the nth success occurs at stage k, then $\mathcal{G}_n = \mathcal{F}_k$.

The problem for the theorem of Bruss deals with the special case where the X_j are independent and $\mathcal{F}_j$ is equal to $\sigma(X_1, \ldots, X_j)$. This means that nothing other than the X_j's are observed.

Let us find the 1-stage look-ahead rule for the dependent case with the Sakaguchi extension. Suppose we are at stage k and that $X_k = 1$. If we stop at this stage, the probability we have selected the last success is

$$\begin{aligned} V_k &= \mathrm{P}(\text{there are no successes after stage } k \,|\mathcal{F}_k) \\ &= \mathrm{P}(X_{k+1} = X_{k+2} = \cdots = 0 \,|\mathcal{F}_k). \end{aligned} \tag{9}$$

This is the denominator of the ratio in (8). If we continue and stop at the next $j > k$ for which $X_j = 1$, if any, our expected return is $W_k + \omega V_k$, where

$$
\begin{aligned}
W_k &= \mathrm{P}(\text{there is exactly one success after stage } k \,|\mathcal{F}_k) \\
&= \sum_{j=k+1}^{\infty} \mathrm{P}(X_{k+1} = \cdots = X_{j-1} = 0, X_j = 1, X_{j+1} = X_{j+2} = \cdots = 0 \,|\mathcal{F}_k),
\end{aligned} \tag{10}
$$

the numerator of the ratio in (8). Thus, the 1-sla is

$$
N_1 = \min\{k \geq 1 : X_k = 1 \text{ and } \frac{W_k}{V_k} < 1 - \omega\}. \tag{11}
$$

The problem is monotone if the following condition is satisfied: If the 1-sla calls for stopping at some stage j with $X_j = 1$, then at any future stage k with $X_k = 1$ the 1-sla will also call for stopping no matter what else is observed (a.s.). This means that if $X_j = 1$ and $W_j/V_j \leq 1-\omega$, then at the next k for which $X_k = 1$, we will a.s. have $W_k/V_k \leq 1-\omega$.

In particular, the problem is monotone if

$$
\frac{W_k}{V_k} \text{ is a.s. nonincreasing in } k. \tag{12}
$$

For the purposes of most applications, it will suffice to check condition (12). Only in special cases, as in Section 5, will it be useful to take advantage of the weaker condition of the previous paragraph. The following corollary now follows immediately from Theorem 1.

Corollary 1. *Suppose the Bernoulli variables $X_1, X_2, \ldots$ satisfy the condition that there are a finite number of successes with probability one. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be an increasing sequence of σ-fields such that $\{X_j = 1\}$ is in $\mathcal{F}_j$ for all j. Then among stopping rules adapted to the sequence $\{\mathcal{F}_j\}$, the rule N_1 of (11) is an optimal stopping rule provided condition (12) is satisfied.*

This may be considered as a Sum-the-Odds Theorem in the sense that the ratio, W_k/V_k in (11), may be written as $\sum_{j=k+1}^{\infty} P_{jk}/(1-P_{jk})$, where

$$
P_{jk} = \mathrm{P}(X_j = 1|\mathcal{F}_k, X_{k+1} = \cdots = X_{j-1} = 0, X_{j+1} = X_{j+2} = \cdots = 0).
$$

It is easy to see that this corollary implies the theorem of Bruss. In the theorem of Bruss, the X_j are independent and $\mathcal{F}_j = \sigma(X_1, \ldots, X_j)$. So the conditioning in the definition of P_{jk} may be ignored, and $P_{jk} = p_j$. Then, assuming $\sum_{i=1}^{\infty} p_i < \infty$, we have

$$
\frac{W_k}{V_k} = \sum_{j=k+1}^{\infty} \frac{p_j}{1-p_j} \tag{13}
$$

so that in this case, $N_1 = N^*$ of (2). We see from this that the W_k/V_k are non-random and nonincreasing, so that (12) is satisfied. Thus the problem is monotone and the 1-sla, N^*, is optimal. This proves the result of Bruss in the infinite horizon case, and contains the Sakaguchi extension.

4. Full-Information Best-Choice Problems. In full-information best-choice problems, independent random variables, $Y_1, Y_2, \ldots$, with known continuous distribution functions, $F_1(y), F_2(y), \ldots$, respectively, are observed sequentially. It is desired to choose a stopping rule that maximizes the probability of stopping on the largest observation. By the Kolmogorov zero-one law, the probability that there is a largest observation is either zero or one. If there is no largest observation, then the task is impossible. Therefore we assume that with probability one there is a largest observation.

Let $M_k = \max\{Y_1, \ldots, Y_k\}$ be the maximum of the first k observations. The Bernoulli variables of the preceding section are therefore $X_1, X_2, \ldots$, where $X_k = \mathrm{I}(Y_k = M_k)$. The problem is to stop on the last success, that is, the last record value.

In (9) and (10), the sigma-field $\mathcal{F}_k$ is the sigma-field generated by the variables, $Y_1, \ldots, Y_k$. We may compute

$$V_k = \prod_{i=k+1}^{\infty} F_i(M_k) \tag{14}$$

and

$$W_k = \sum_{j=k+1}^{\infty} \left[\prod_{i=k+1}^{j-1} F_i(M_k) \right] \int_{M_k}^{\infty} \left[\prod_{i=j+1}^{\infty} F_i(y) \right] dF_j(y) \tag{15}$$

We may therefore write

$$\frac{W_k}{V_k} = \sum_{j=k+1}^{\infty} \int_{M_k}^{\infty} \left[\prod_{i=j+1}^{\infty} F_i(y) \right] dF_j(y) / \prod_{i=j}^{\infty} F_i(M_k). \tag{16}$$

It is easy to see that W_k/V_k is a.s. nonincreasing. The first term of the sum on the right is never negative, so removing it does not increase the sum. M_k is nondecreasing a.s. so the range of the integral never increases. Finally, the term, $\prod_{i=j}^{\infty} F_i(M_k)$ is a.s. nondecreasing in k, so its reciprocal is a.s. nonincreasing. Hence the 1-sla, N_1 of (11), is optimal.

Theorem 2. *For the full-information best-choice problem with independent observations, the one-stage look-ahead rule is optimal.*

4.1 The full-information best-choice problem with i.i.d. observations. As an example, consider the problem, solved in Gilbert and Mosteller (1966), when there are a *finite* number of independent observations having the same continuous distribution, which may be taken without loss of generality to be the uniform distribution on the interval (0,1), $F_j(y) = y$ for $0 < y < 1$ and for all j. Let n denote the number of observations.

Equation (16) becomes

$$
\begin{aligned}
\frac{W_k}{V_k} &= \sum_{j=k+1}^{n} \int_{M_k}^{1} y^{n-j}\, dy \frac{1}{M_k^{n-j+1}} \\
&= \sum_{j=k+1}^{n} \frac{1}{n-j+1}(1-M_k^{n-j+1})\frac{1}{M_k^{n-j+1}}
\end{aligned}
\tag{17}
$$

Therefore, the optimal rule of (11) may be written

$$
N_1 = \min\{k \geq 1 : Y_k = M_k \text{ and } \sum_{j=k+1}^{n} \frac{1}{n-j+1}\left(\frac{1}{M_k^{n-j+1}} - 1\right) < 1-\omega\}. \tag{18}
$$

Gilbert and Mosteller, under the condition that $\omega = 0$, state the optimal rule in a different form: Stop at the first k for which $Y_k = M_k$ and $M_k > b_{n-k}$, where $b_0 = 0$ and for $m \geq 1$, b_m is the root, b, of the equation

$$
\sum_{j=1}^{m} \frac{1}{j}\binom{m}{j}\left(\frac{1}{b} - 1\right)^{j} = 1-\omega \tag{19}
$$

between 0 and 1. (Note: Gilbert and Mosteller use the notation b_{m+1} for this root.) Here m represents the number of observations remaining; if your present observation is a record value, then you should stop if and only if it is greater than b_m.

When the optimal rule (18) is put into this form, it becomes: If your present observation is a record value and if there are m observations remaining, then you should stop if it is greater than b_m, where b_m is the root, b, of the equation

$$
\sum_{j=1}^{m} \frac{1}{j}\left(\frac{1}{b^j} - 1\right) = 1-\omega. \tag{20}
$$

This is a somewhat simpler equation than (19), but these two equations must be equivalent since they give the same optimal strategy. Perhaps the simplest way to check they are the same is to replace $1/b$ in both equations by x, note that the left sides are equal at $x = 1$ and that their derivatives with respect to x are both equal to $(x^m - 1)/(x-1) = \sum_{i=0}^{m-1} x^i$.

4.2 The full-information best-choice problem with batch arrivals. As an example in which the variables have different distributions, suppose that for $i = 1, 2, \ldots$, Y_i has the beta distribution with distribution function and density

$$
F_i(y) = y^{\theta_i} \quad \text{and} \quad f_i(y) = \theta_i y^{\theta_i - 1} \text{ for } 0 < y < 1, \tag{21}
$$

where $\theta_1, \theta_2 \ldots$ are given positive numbers. We assume $\sum_1^\infty \theta_i < \infty$ so that from the Borel-Cantelli Lemma, there will be a finite number of record values with probability one.

This example provides an extension of the no-information best-choice problem with batch arrivals of Hsiau and Yang (2000) to the full-information case. In this problem, candidates arrive in batches, with $\ell_i \geq 1$ candidates arriving on day i for $i = 1, \ldots, n$, with n finite. All candidates arriving on day i are interviewed together, and the best among them may be considered as the observation for day i. Each candidate has a value chosen i.i.d. from a known continuous distribution, which may be taken without loss of generality to be the uniform distribution on (0,1). The distribution of the observation, Y_i for the ith day, is the distribution of the maximum of a sample of size ℓ_i from the uniform distribution on (0,1). The leads to the beta distribution given in (21) with $\theta_i = \ell_i$ for $i = 1, \ldots, n$.

Let us find the optimal rule of (11) for the problem with distributions (21). Let $s_j = \sum_{i=j}^{\infty} \theta_i$. From (16),

$$\begin{aligned} \frac{W_k}{V_k} &= \sum_{j=k+1}^{\infty} \int_{M_k}^{1} y^{s_j - 1}\, dy \frac{1}{M_k^{s_j}} = \sum_{j=k+1}^{\infty} \frac{1}{s_j} [1 - M_k^{s_k}] \frac{1}{M_k^{s_j}} \\ &= \sum_{j=k+1}^{\infty} \frac{M_k^{-s_j} - 1}{s_j} \end{aligned} \tag{22}$$

so the optimal rule is, similar to (18),

$$N_1 = \min\{k \geq 1 : Y_k = M_k \text{ and } \sum_{j=k+1}^{\infty} \frac{M_k^{-s_j} - 1}{s_j} < 1 - \omega\}. \tag{23}$$

This stopping rule may also be described in a manner analogous to (20) as follows. If at stage k the present observation, Y_k, is a record value, then stop if $Y_k (= M_k)$ is greater than c_k, where c_k is the unique root, c, of the equation

$$\sum_{j=k+1}^{\infty} \frac{1}{s_j} \left(\frac{1}{c^{s_j}} - 1 \right) = 1 - \omega, \tag{24}$$

similar to (20). Equation (19) has no such simple analog for this problem.

5. Sum-the-Odds Theorem: Positive Dependence. In the positive dependent case, an observation of a success (resp. failure) on a trial increases the probability of success (resp. failure) on future trials. To illustrate the difficulties that arise with positive dependence, consider the problem where the conditional distribution of $X_1, \ldots, X_n$ given p are i.i.d. Bernoulli(p), and the prior distribution of p is the beta distribution, $Be(\alpha, \beta)$. As is well known, the posterior distribution after observing $X_1, \ldots, X_k$ is $Be(\alpha + S_k, \beta + k - S_k)$, where $S_k = \sum_1^k X_i$ is the number of successes in the first k trials.

Let us compute W_k/V_k first for $k = 0$. We find

$$V_0 = \mathrm{E}(1-p)^n = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha)\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} = \frac{\Gamma(\alpha+\beta)\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(\alpha+\beta+n)} \tag{25}$$

and

$$W_0 = \mathrm{E}(np(1-p)^{n-1}) = \frac{n\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta+n-1)}{\Gamma(\alpha+\beta+n)} = \frac{n\alpha\Gamma(\alpha+\beta)\Gamma(\beta+n-1)}{\Gamma(\beta)\Gamma(\alpha+\beta+n)} \tag{26}$$

Therefore,

$$\frac{W_0}{V_0} = \frac{n\alpha}{n+\beta-1}. \tag{27}$$

It follows that for arbitrary $k \le n$,

$$\frac{W_k}{V_k} = \frac{(n-k)(\alpha+S_k)}{n+\beta-S_k-1}. \tag{28}$$

The 1-sla is

$$N_1 = \min\{k : X_k = 1 \text{ and } \frac{(n-k)(\alpha+S_k)}{n+\beta-S_k-1} < 1\}. \tag{29}$$

In general, the 1-sla is not monotone because, even though W_k/V_k decreases with each failure, it may increase with a success. We now search for conditions under which W_k/V_k decreases with a success. If $W_k/V_k \ge 1$ (and $X_k = 1$), the 1-sla calls for continuing so it is optimal to continue. So we may assume that $W_k/V_k < 1$ and find a condition under which $W_{k+1}/V_{k+1} < 1$ when $X_{k+1} = 1$. From (28) we may write the inequality $W_k/V_k < 1$ as

$$(n-k)(\alpha+S_k) < n+\beta-S_k-1 \tag{30}$$

and we seek conditions under which

$$(n-k-1)(\alpha+S_k+1) < n+\beta-S_k-2. \tag{31}$$

Using (30), we have

$$\begin{aligned}(n-k-1)(\alpha+S_k+1) &= (n-k)(\alpha+S_k)+(n-k)-(\alpha+S_k)-1\\ &< (n+\beta-S_k-1)+(n-k)-(\alpha+S_k)-1.\end{aligned} \tag{32}$$

Thus (31) is satisfied if

$$n-k \le \alpha+S_k. \tag{33}$$

We may conclude that the 1-sla acts optimally if, at the first k for which $X_k = 1$ and (30) is satisfied, (33) is also satisfied. In particular, if $\alpha \ge n-2$, the 1-sla is optimal.

As an example, suppose $n = 12$, $\alpha = 2$ and $\beta = 2$. Then

$$N_1 = \min\{k : X_k = 1 \text{ and } \frac{(12-k)(2+S_k)}{n-S_k+1} \le 1\}. \tag{34}$$

Suppose seven failures are followed by a success, so that $k = 8$, $S_8 = 1$ and $X_8 = 1$. Then $W_8/V_8 = 12/12 = 1$, so that the 1-sla is indifferent between stopping and continuing.

Yet if $X_9 = 1$, then $W_9/V_9 = 12/11 > 1$ and it is strictly optimal to take another observation. Thus the 1-sla can be improved. (To be more precise, we should take $\beta = 2+\epsilon$; then for sufficiently small positive ϵ, the 1-sla calls for stopping at stage k (it is no longer indifferent), while the improved value of continuing has not changed significantly.)

It is not hard to show that for any other sequence of successes and failures in this example the 1-sla acts optimally.

6. Sum-the-Odds Theorem: Negative Dependence. To illustrate the problems that arise with negative dependence, consider the case where n balls are drawn without replacement from an urn containing a red balls and b blue balls, where $a+b \geq n$. We wish to find a stopping rule to maximize the probability of stopping on the last red ball drawn. So we take $X_i = \mathrm{I}\{i\text{th draw is red}\}$. Let a_k and b_k denote the number of red and black balls respectively remaining after k balls have been observed, starting with $a_0 = a$ and $b_0 = b$. If $b_k < n-k$ at stage k, then there is bound to be another red in the remaining $n-k$ draws, so one should not stop. Assuming $b_k \geq n-k$, V_k and W_k may be computed using the hypergeometric distribution.

$$V_k = \mathrm{P}\{\text{the next } (n-k) \text{ draws are blue} \,|a_k, b_k\} = \frac{\binom{b_k}{n-k}}{\binom{a_k+b_k}{n-k}} \tag{35}$$

and

$$W_k = \mathrm{P}\{1 \text{ red and } (n-k-1) \text{ blue} \,|a_k, b_k\} = \frac{\binom{a_k}{1}\binom{b_k}{n-k-1}}{\binom{a_k+b_k}{n-k}} \tag{36}$$

from which we may compute

$$\frac{W_k}{V_k} = \frac{a_k(n-k)}{b_k-n+k+1}. \tag{37}$$

We now show that W_k/V_k is almost surely decreasing in k. If $X_{k+1} = 0$, then $W_{k+1}/V_{k+1} = a_k(n-k-1)/(b_k-n+k+1)$ which is smaller than W_k/V_k. If $X_{k+1} = 1$, then $W_{k+1}/V_{k+1} = (a_k-1)(n-k-1)/(b_k-n+k+2)$ which is also smaller than W_k/V_k. Thus the 1-sla is optimal. It is the stopping rule

$$N_1 = \min\{k \leq n : X_k = 1 \text{ and } (a_k+1)(n-k) < b_k+1\}. \tag{38}$$

This result should be true in greater generality, but there seems to be many ways to try to extend it. One way is to assume that negative dependence arises in the following way. The probabilities p_k are determined by numbers a and b, and by two sequences $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$, not necessarily integers. We assume the urn has initially a red and b blue balls. On the appearance of the jth red ball (resp. jth blue ball), α_j red balls (resp. β_j

blue balls) are removed from the urn. We assume that $a \geq \sum_1^n \alpha_j$ and $b \geq \sum_1^n \beta_j$. It can be shown that if $\max \alpha_j \leq \min \beta_j$, then the 1-sla is optimal. The trouble with this result is that the model is somewhat artificial, and that the whole result should hold in greater generality.

If $\max \alpha_j > \min \beta_j$, then the 1-sla may not be optimal. As an example, suppose all $\beta_i = 0$ and all $\alpha_i = 0$ except $\alpha_2 = a$. Let $p = a/(a+b)$ and $q = 1-p$. Then $V_0 = q^n$ and $W_0 = npq^{n-1}$. Assuming the possibility of stopping at stage 0 (with $X_0 = 1$), the 1-sla calls for stopping if $W_0 \leq V_0$, or equivalently if $p \leq n/(n+1)$. Yet if we observe $X_1 = 1$, then $V_1 = q^{n-1}$ and $W_1 = p + pq + pq^2 + \cdots + pq^{n-2} = 1 - q^{n-1}$, and we find that the 1-sla still calls for stopping if and only if $W_1 \leq V_1$, or equivalently, if $q^{n-1} \geq 1/2$. If $p = n/(n+1)$ or slightly less, the 1-sla calls for stopping at the initial stage, yet $q^{n-1} = (n/(n+1))^{n-1}$ which for large n is close to $e^{-1} < 1/2$ so for large n ($n \geq 5$ suffices) the 1-sla calls for continuing. So the 1-sla is not optimal.

7. On a Stopping Game of Sakaguchi. Let $X_1, X_2, \ldots$ be a sequence of independent Bernoulli random variables and let $p_i = \mathrm{P}(X_i = 1)$ be the probability of success. We assume $\sum_1^\infty p_i < \infty$ so that with probability one there are a finite number of successes. For simplicity, we assume that $p_i < 1$ for all i. Two players sequentially observe the X_i and vie with each other to predict the last success. As each X_i is observed, Player I is given priority whether or not to predict the present observation as the last success. If Player I chooses not to make the prediction, then Player II is given the option. It is assumed that the p_i are known to both players. The player who makes the prediction wins if his prediction is true and loses if it is false. If neither player makes a prediction, the game is called a tie. We take the payoff from Player I's point of view to be one for a win, zero for a loss and ω for a tie. Here it is not assumed that $\omega < 1$, since the problem is non-trivial even if $\omega \geq 1$.

Sakaguchi (1984) solved this problem with $p_i = 1/i$ for $i = 1, \ldots, n$ and $p_i = 0$ for $i > n$ as in the secretary problem, and with $\omega = 1/2$.

This is a multistage game of perfect information. If it were a finite game, it could be solved by backward induction. However, a simple observation changes it into a finite horizon problem. Let V_j denote the probability that no successes occur after stage j,

$$V_j = \prod_{i=j+1}^{\infty} (1 - p_i) \tag{39}$$

for all j. Note that the V_j are nondecreasing and converging to 1 as $j \to \infty$. If $X_j = 1$ and $V_j \geq 1/2$, then it is clearly optimal for Player I to stop at j, since his probability of win is at least 1/2, and if he passes, his opponent can stop and win with probability at least 1/2. Let

$$m = \min\{j : V_j \geq 1/2\}. \tag{40}$$

We know what will happen if play reaches stage m: Player I will stop at the next success if any, since if he doesn't, Player II will. If stage m is reached, then Player I's expected payoff is

$$
\begin{aligned}
U_m &= \mathrm{P}(\text{exactly one success from } m \text{ on}) + \omega \mathrm{P}(\text{no successes from } m \text{ on}) \\
&= \sum_{j=m}^{\infty} \Big[\prod_{i=m}^{j-1}(1-p_i)\Big] p_j V_j + \omega V_{m-1} = V_{m-1}\Big[\sum_{j=m}^{\infty} \frac{p_j}{1-p_j} + \omega\Big] \\
&= V_{m-1}\Big[\sum_{j=m}^{\infty} r_j + \omega\Big]
\end{aligned} \tag{41}
$$

where $r_j = p_j/(1-p_j)$ is the odds ratio at stage j. Thus the game may be reformulated to be: The payoff to Player I is V_j if he stops at j, and $1-V_j$ if Player II stops at j for $j = 1, \ldots, m-1$. If neither player has stopped by stage m, the payoff to Player I is U_m.

8.1 A more general model. Although one could solve this game by the method used by Sakaguchi, it is simpler and gives more insight to consider this game in a slightly generalized form as a multistage game of perfect information. Let $0 < V_1 < V_2 < \cdots < V_{m-1} < 1/2$ be given numbers, and let $p_1, \ldots, p_{m-1}$ be given probabilities, not necessarily related to the V_i by (39). Also, let U_m be an arbitrary number, not necessarily given by (41). The game at stage k for $k = 1, \ldots, m-1$, denoted by G_k, may be described as follows. The first move is a chance move with success probability p_k. If failure occurs, the game moves on the game G_{k+1}. If success occurs, Then Player I may stop and receive V_k, or continue. If Player I continues, Player II may stop giving Player I the payoff $1 - V_k$, or continue. If Player II continues, the game moves on to the game G_{k+1}. The game G_m is simply the game that gives Player I the amount U_m. In symbols,

$$
\begin{aligned}
G_k &= p_k \left[\begin{array}{c c} & \begin{array}{cc} \text{stop} & \text{wait} \end{array} \\ \begin{array}{c} \text{stop} \\ \text{wait} \end{array} & \begin{pmatrix} V_k & V_k \\ 1-V_k & G_{k+1} \end{pmatrix} \end{array} \right] + (1-p_k) G_{k+1} \quad \text{for } k = 1, 2, \ldots, m-1 \\
G_m &= (U_m).
\end{aligned} \tag{42}
$$

The value of G_m is U_m obviously. Let U_k denote the value of G_k. The U_k may be found by backward recursion using a well-known technique in multistage games. (See for example Ferguson (2005), Part II Chapter 6.2.)

$$
U_k = p_k \,\mathrm{Val} \begin{pmatrix} V_k & V_k \\ 1-V_k & U_{k+1} \end{pmatrix} + (1-p_k) U_{k+1} \tag{43}
$$

for $k = m-1, m-2, \ldots, 1$. The game inside the Val sign always has a saddle point, and since $V_k < 1 - V_k$ for $k < m$, we have

$$
\mathrm{Val} \begin{pmatrix} V_k & V_k \\ 1-V_k & U_{k+1} \end{pmatrix} = \begin{cases} V_k & \text{for } U_{k+1} \le V_k \\ U_{k+1} & \text{if } V_k \le U_{k+1} \le 1 - V_k \\ 1 - V_k & \text{if } 1 - V_k \le U_{k+1}. \end{cases} \tag{44}
$$

Therefore,

$$U_k = \begin{cases} p_k V_k + (1-p_k)U_{k+1} & \text{for } U_{k+1} \le V_k \\ U_{k+1} & \text{if } V_k \le U_{k+1} \le 1 - V_k \\ p_k(1-V_k) + (1-p_k)U_{k+1} & \text{if } 1 - V_k \le U_{k+1}. \end{cases} \tag{45}$$

Note that if $U_m \le 1/2$, the U_k are constant or increasing as k gets smaller. Similarly for $U_m \ge 1/2$, the U_k are constant or decreasing as k gets smaller. This implies that there are two types of optimal behavior depending U_m. If $U_m \ge 1/2$, Player I should continue up to m and Player II may stop before. If $U_m \le 1/2$, Player II should continue up to m and Player I may stop before.

Let $k_1 = \max\{k : U_k > V_{k-1}\}$ and $k_2 = \max\{k : U_k < 1 - V_{k-1}\}$.

Theorem 3. *The value of the game is U_1. If $U_m \le 1/2$, it is optimal for Player I to stop at the first success from stage k_1 on and for Player II to continue up to m. If $U_m \ge 1/2$, it is optimal for Player II to stop at the first success from stage k_2 on and for Player I to continue up to m.*

8.2 Application to Sakaguchi's Game. Now consider the generalization of the game of Sakaguchi in which V_j is related to the p_i by (39), m is given by (40), and U_m is given by (41). We treat the two cases of Theorem 3 separately.

Suppose first that $U_m \le 1/2$. Player II's optimal behavior is clear: stop at the first success from stage m on if given the chance. Player I will certainly stop from stage m on, and he may stop earlier. He will stop at stage $m-1$ if $V_{m-1} \ge U_m$, which reduces to $\sum_{j=m}^{\infty} r_j \le 1 - \omega$. This is the behavior entailed in the optimal stopping rule N^* of (2). We may compute U_{m-1} from (41) and the top line of (45) as

$$\begin{aligned} U_{m-1} &= p_{m-1}V_{m-1} + (1-p_{m-1})U_m = V_{m-1}[p_{m-1} + (1-p_{m-1})\Big[\sum_{j=m}^{\infty} r_j + \omega\Big] \\ &= V_{m-2}\Big[\sum_{j=m-1}^{\infty} r_j + \omega\Big] \end{aligned} \tag{46}$$

Therefore at stage $m-2$, Player I will stop if $\sum_{j=m-1}^{\infty} r_j \le 1-\omega$. Note that (46) is just (41) with m replaced by $m-1$. Therefore, this analysis continues by induction down to stage

$$k_1 = \min\{k : \sum_{j=k+1}^{\infty} r_j \le 1 - \omega\}.$$

This is the same cutoff point used by the stopping rule N^* of (2)! However in this game, Player I should stop from stage m on even if $\sum_{j=m+1}^{\infty} r_j > 1-\omega$. Thus, Player I's optimal strategy is to stop at the first success from stage $\min\{k_1, m\}$ on, while Player II's optimal strategy is to stop at the first success from stage m on.

Now, suppose $U_m \geq 1/2$. (The problem treated by Sakaguchi, where $p_i = 1/i$ for $i \leq n$, $p_i = 0$ for $i > n$ and $\omega = 1/2$, falls in this case.) The computation corresponding to (46) is only slightly more complex. Let θ denote $\sum_{j=m}^{\infty} r_j + \omega$, so that $U_m = V_{m-1}\theta$. From (45), we evaluate U_{m-1} assuming $1 - V_{m-1} < U_m$.

$$\begin{aligned} U_{m-1} &= p_{m-1}(1 - V_{m-1}) + (1 - p_{m-1})V_{m-1}\theta \\ &= p_{m-1}V_{m-1}(\frac{1}{V_{m-1}} - 1) + V_{m-2}\theta \\ &= V_{m-2}[r_{m-1}(\frac{1}{V_{m-1}} - 1) + \theta] \end{aligned} \tag{47}$$

This analysis may be repeated to find U_k for $k < m - 1$ as long as $1 - V_k \leq U_{k+1}$. We find

$$U_k = V_{k-1}[r_k(\frac{1}{V_k} - 1) + \cdots + r_{m-1}(\frac{1}{V_{m-1}} - 1) + \theta]. \tag{48}$$

This stops at

$$k_2 = \min\{k : \sum_{j=k+1}^{m-1} r_j(\frac{1}{V_j} - 1) + \sum_{j=m}^{\infty} r_j \leq 1 - \omega\}. \tag{49}$$

In this case, Player II's optimal strategy is to stop at the first success from stage $\min\{k_2, m\}$ on, while Player I's optimal strategy is to stop at the first success from stage m on.

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Thomas S. Ferguson
Department of Mathematics
University of California at Los Angeles
Los Angeles, CA 90095
tom@math.ucla.edu