Large Sample Theory
Ferguson

Exercises, Section 8, The Sample Correlation Coefficient.

1. Use the notation $\mu_{ij} = \mathrm{E}(X-\mathrm{E}X)^i(Y-\mathrm{E}Y)^j$ to represent the ij^{th} central product moment. Note that $\mu_{20} = \sigma_x^2$, $C(XX,XX) = \mu_{40} - \sigma_x^4$, etc. Show that the limiting variance of the sample correlation coefficient is

$$\gamma^2 = \frac{\rho^2}{4}\left[\frac{\mu_{40}}{\sigma_x^4} + 2\frac{\mu_{22}}{\sigma_x^2\sigma_y^2} + \frac{\mu_{04}}{\sigma_y^4}\right] - \rho\left[\frac{\mu_{31}}{\sigma_x^3\sigma_y} + \frac{\mu_{13}}{\sigma_x\sigma_y^3}\right] + \frac{\mu_{22}}{\sigma_x^2\sigma_y^2}$$

2. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a sample of size n from a distribution on the plane with finite fourth moments.
(a) Find the asymptotic distribution of $Z_n = \log(s_x^2/s_y^2)$.
(b) Find an asymptotically distribution-free confidence interval for $\theta = \log(\sigma_x^2/\sigma_y^2)$.

3. Find the variance-stabilizing transformations for $\overline{X}_n$ when sampling from
(a) the gamma distribution, $\mathcal{G}(\alpha, 1)$, with probability density $f(x|\alpha) = x^{\alpha-1}e^{-x}/\Gamma(\alpha)$ on the interval $(0,\infty)$,
(b) the geometric distribution, $\mathrm{P}_\theta(X = x) = (1-\theta)\theta^{x-1}$ for $x = 1, 2, \ldots$.

4. (a) In testing the hypothesis H_0: $\rho = 0$, we may use the test that rejects H_0 if $|r|$ is too large. Find the asymptotic distribution of $\sqrt{n}\,r$ when $\rho = 0$ and note that it is not asymptotically distribution-free under H_0 (within the class of distribution with finite fourth moments).
(b) In testing H_0: X and Y are independent, show that the test that rejects H_0 if $|r|$ is too large is asymptotically distribution-free under H_0 (within the class of distribution with finite fourth moments).