Solutions to the Exercises of Section 1.7.

1.7.1(a) The minimax point is the intersection of the line joining $(-2,3)$ to $(-3/4,-9/4)$ and the line $x=y$. The former line has slope $-21/5$ and equation

$$y-3=(-21/5)(x+2).$$

Putting $y=x$ and solving yields $x=y=-27/26$. Therefore the minimax point is $(-27/26,-27/26)$. The minimax rule is the randomized rule that mixes d_1 (associated with the point $(-2,3)$) and d_2 (associated with $(-3/4,-9/4)$) with probababilities p and $1-p$ with p chosen so that

$$(-27/26,-27/26)=p(-2,3)+(1-p)(-3/4,-9/4).$$

Solving for p gives $p=3/13$. The minimax risk is $-27/26$.

(b) Since the line on which the minimax point lies has slope $-21/5$, the prior distribution $(p,1-p)$ with respect to which the minimax rule is Bayes is a vector with slope 5/21. Solving

$$(1-p)/p=5/21$$

for p yields $p=21/26$. This prior distribution, θ_1 w.p. 21/26 and θ_2 w.p. 5/26, is also least favorable.

(c) The prior distribution (1/2,1/2) is perpendicular to lines of slope -1, so we search for a point on the lower boundary such that the line with slope -1 through this point is tangent to the risk set. Since the line joining $(-2,3)$ to $(-3/4,-9/4)$ has slope $-27/26<-1$ and the slope of the line joining $(-3/4,-9/4)$ to $(3,-4)$ is $-7/15>-1$, the point $(-3/4,-9/4)$ is the Bayes point and the nonrandomized rule d_2 is Bayes with respect to $(1/2,1/2)$. The minimum Bayes risk is $(1/2)(-3/4)+(1/2)(-9/4)=-2$.

1.7.2. Let $\mathcal{C}$ be any collection of convex sets in some space, and let

$$A=\bigcap_{C\in\mathcal{C}} C.$$

To show A convex, let $x\in A$, $y\in A$, and $0\le\alpha\le 1$, and try to show that $\alpha x+(1-\alpha)y\in A$. Since $x\in A$ and $y\in A$, we have for every $C\in\mathcal{C}$ that $x\in C$ and $y\in C$. Then since every $C\in\mathcal{C}$ is convex, we have $\alpha x+(1-\alpha)y\in C$ for every $C\in\mathcal{C}$. This implies $\alpha x+(1-\alpha)y\in A$ as was to be shown.

1.7.3. Someone tosses a fair coin until the first tails appears and announces X, the outcome of the experiment, but neglects to tell you whether X is the number of heads observed ($\theta=1$), or the total number of tosses ($\theta=0$). The set of possible outcomes he may announce is $\mathcal{X}=\{0,1,2,\ldots\}$. You must decide whether $\theta=0$ or $\theta=1$ with zero/one loss. The set of nonrandomized decision rules is $D=\{d: d(x)=0 \text{ or } 1, \text{ for } x=0,1,\ldots\}$. To find the risk set, S, we first find the nonrandomized risk set S_0, and then find its convex hull. For a given rule, d, the risk point is $(R(0,d),R(1,d))$ where

$$\begin{aligned}R(0,d)&=0\cdot P(d(X)=0|\theta=0)+1\cdot P(d(x)=1|\theta=0)\\&=P(d(X)=1|\theta=0)=\sum_{\substack{x\ge1\\ d(x)=1}}2^{-x}\\R(1,d)&=P(d(X)=0|\theta=1)=\sum_{\substack{x\ge0\\ d(x)=0}}2^{-(x+1)}\end{aligned}$$

As important examples, the four rules,

$$\begin{aligned}&d_1(x)=1 &&\text{for all } x.\\&d_2(x)=1 &&\text{for } x=0 \text{ and } d_2(x)=0 \text{ for all } x>0.\\&d_3(x)=0 &&\text{for } x=0 \text{ and } d_3(x)=1 \text{ for all } x>0.\\&d_4(x)=0 &&\text{for all } x.\end{aligned}$$

have risk points (1,0), (0,1/2), (1,1/2), and (0,1), respectively. For a general rule d, we distinguish two cases: if $d(0) = 0$, then $R(1,d) = 1/2 + (1/2)(1 - R(0,d))$, so that the point $(R(0,d), R(1,d))$ falls on the line segment from (0,1) to (1,1/2); if $d(0) = 1$, then $R(1,d) = (1/2)(1 - R(0,d))$, so that the point $(R(0,d), R(1,d))$ falls on the line segment from (0,1/2) to (1,0). The risk set therefore consists of the parallelogram between the four points (0,1/2), (1,0), (1,1/2), and (0,1). The minimax point lies at the intersection of the line from (0,1/2) to (1,0) (equation $y = (-1/2)(x - 1)$), and the diagonal line $x = y$, namely (1/3,1/3). The minimax rule chooses d_1 with probability p and d_2 with probability $1 - p$, where

$$p \cdot (1,0) + (1-p) \cdot (0,1/2) = (1/3,1/3).$$

Solving for p gives $p = 1/3$. Since the slope of the line from (1,0) to (0,1/2) is $-1/2$, the least favorable distribution chooses $\theta = 0$ w.p. q, and $\theta = 1$ w.p. $1 - q$, where

$$(1-q)/q = 2.$$

Solving for q gives $q = 1/3$. There is in fact a nonrandomized minimax rule, namely the rule, d^*, that chooses $\theta = 1$ if X is even and $\theta = 0$ if X is odd:

$$R(0,d^*) = 1/4 + 1/16 + 1/64 + \cdots = 1/3,$$
$$R(1,d^*) = (1/2)(1 - 1/3) = 1/3.$$

1.7.4. The risk set S consists of the line segment joining the point $((2P-1)(a+b), (2P-1)a + Pb)$ to the point $(a, (2P-1)a)$. The first point is above the line $x = y$, and the second point is below. There are two cases, depending on whether the slope of the line is positive or negative. If $P > (2a+b)/(2a+2b)$, the slope of the line is positive and the minimax point is the lower point, $(a, (2P-1)a)$ with minimax value, a. This corresponds to the "fold" strategy of player II. The maximin choice for player I is the "bluff" strategy, which gives him the first coordinate, value a, which is greater than the second coordinate, value $(2P-1)a$.

If $P < (2a+b)/(2a+2b)$, the slope of the line is negative and the minimax point will be the point on this line with $x = y$. This line has slope $m = -Pb/(a - (2P-1)(a+b))$ and equation $y - (2P-1)a = m(x-a)$. Putting $x = y$ and solving gives

$$x = (2P - 1 - m)a/(1-m) = a(4(a+b)P - (2a+b))/(2a+b)$$

as the minimax value. The minimax strategy of II chooses "call" with probability p and "fold" with probability $1 - p$, where p is chosen so that

$$p(2P-1)(a+b) + (1-p)a = x.$$

This gives

$$p = (x-a)/((2P-1)(a+b) - a) = 2a/(2a+b).$$

It is interesting to note that this probability is independent of P; player II does not need to know the probability that player I has a winning card in order to play optimally (provided $P < (2a+b)/(2a+2b)$.) To find player I's maximin strategy, we set $(1-q)/q$ equal to the negative of the slope, $(1-q)/q = -1/m$, and solve for q:

$$q = Pb/((1-P)(2a+b)).$$

Player I should use the "bluff" strategy with probability q and the "honest" strategy with probability $1-q$.