Solutions to the Exercises of Section 2.2.

2.2.1. Since the distribution τ degenerate at θ has risk $r(\tau,\delta) = R(\theta,\delta)$, we have $\sup_{\tau\in\Theta^*} r(\tau,\delta) \geq \sup_{\theta\in\Theta} R(\theta,\delta)$. Suppose that $\sup_{\tau\in\Theta^*} r(\tau,\delta) > \sup_{\theta\in\Theta} R(\theta,\delta)$. This implies there exists a τ such that $R(\theta,\delta) < r(\tau,\delta)$ for all $\theta \in \Theta$. But $r(\tau,\delta)$ is the expectation of $R(\theta,\delta)$ over θ using the distribution τ and so is also less than $r(\tau,\delta)$, a contradiction that completes the proof.

2.2.2. We are given that the game has a value,

$$\inf_\delta \sup_\tau r(\tau,\delta) = \sup_\tau \inf_\delta r(\tau,\delta),$$

that τ_0 is least favorable,

$$\inf_\delta r(\tau_0,\delta) = \sup_\tau \inf_\delta r(\tau,\delta),$$

and that δ_0 is minimax,

$$\sup_\tau r(\tau,\delta_0) = \inf_\delta \sup_\tau r(\tau,\delta).$$

We are to show that δ_0 is Bayes with respect to τ_0, i.e. that

$$r(\tau_0,\delta_0) \leq \inf_\delta r(\tau_0,\delta).$$

This follows in one line:

$$r(\tau_0,\delta_0) \leq \sup_\tau r(\tau,\delta_0) = \inf_\delta \sup_\tau r(\tau,\delta). = \sup_\tau \inf_\delta r(\tau,\delta) = \inf_\delta r(\tau_0,\delta).$$

2.2.3. We are given that the game has a value,

$$\inf_\delta \sup_\tau r(\tau,\delta) = \sup_\tau \inf_\delta r(\tau,\delta),$$

and that δ_0 is minimax,

$$\sup_\tau r(\tau,\delta_0) = \inf_\delta \sup_\tau r(\tau,\delta).$$

We are to show that δ_0 is extended Bayes, i.e. we must show that for every $\epsilon > 0$ there exists a prior, τ, such that

$$r(\tau,\delta_0) \leq \inf_\delta r(\tau,\delta) + \epsilon.$$

The proof is by contradiction. Suppose the conclusion is false. Then there exists an $\epsilon > 0$ such that for every prior τ,

$$r(\tau,\delta_0) > \inf_\delta r(\tau,\delta) + \epsilon.$$

Then, taking the supremum on both sides, we have

$$\sup_\tau r(\tau,\delta_0) \geq \sup_\tau \inf_\delta r(\tau,\delta) + \epsilon = \inf_\delta \sup_\tau r(\tau,\delta) + \epsilon > \inf_\delta \sup_\tau r(\tau,\delta),$$

contradicting the assumption that δ_0 is minimax.