Solutions to the Exercises of Section 3.1.

3.1.1. (a) The joint distribution of X and Y is a mixed discrete and continuous density,

$$f_{X,Y}(x,y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}\binom{n}{x} y^x(1-y)^{n-x}\mathrm{I}(0<y<1) \quad x=0,1,\ldots,n$$

so the marginal distribution of X has mass function,

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\binom{n}{x}\int_0^1 y^{\alpha+x-1}(1-y)^{\beta+n-x-1}\,dy = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\binom{n}{x}\frac{\Gamma(\alpha+x)\Gamma(\beta+n-x)}{\Gamma(\alpha+\beta+n)}$$

for $x=0,1,\ldots,n$, exactly $\mathcal{BB}(\alpha,\beta,n)$.

(b) $\mathrm{E}X = \mathrm{E}(\mathrm{E}(X|Y)) = \mathrm{E}(nY) = n\mathrm{E}Y = n\alpha/(\alpha+\beta)$.

(c) $\mathrm{E}X^2 = \mathrm{E}(\mathrm{E}(X^2|Y)) = \mathrm{E}(\mathrm{Var}(X|Y)+\mathrm{E}(X|Y)^2) = \mathrm{E}(nY(1-Y)+n^2Y^2)$, so

$$\begin{aligned}\mathrm{Var}X &= \mathrm{E}(nY(1-Y)) + \mathrm{Var}(nY) = n(\mathrm{E}Y - \mathrm{E}Y^2) + n^2\mathrm{Var}Y \\ &= n\left(\frac{\alpha}{\alpha+\beta} - \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}\right) + n^2\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \\ &= \frac{n\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}.\end{aligned}$$

3.1.2. It is easier to do this problem in reverse. Let X and Z be independent with binomial distributions $\mathcal{B}(n,p)$ and $\mathcal{B}(M-n,p)$, respectively, and let $Y=X+Z$. We are to show (a) the unconditional distribution of Y is $\mathcal{B}(M,p)$, and (b) the conditional distribution of X given $Y=y$ is $\mathcal{H}(n,y,M)$.

(a) Y is the number of successes in M independent trials with probability p of success on each trial, and so is $\mathcal{B}(M,p)$.

(b) The joint mass function of X and Y

$$f_{X,Y}(x,y) = \binom{n}{x}p^x(1-p)^{n-x}\binom{M-n}{y-x}p^{y-x}(1-p)^{M-n-y+x}$$

for $0\le x\le n$ and $x\le y\le M-n+x$. The conditional mass function of X given $Y=y$ is the ratio of this to $f_Y(y)=\binom{M}{y}p^y(1-p)^{M-y}$, namely,

$$f_{X|Y}(x|y) = \frac{\binom{n}{x}\binom{M-n}{y-x}}{\binom{M}{y}} \qquad \text{for} \quad \max(0,y+n-M)\le x\le \min(y,n).$$

3.1.3. The joint density of Y and Z is

$$f_{Y,Z}(y,z) = \frac{1}{\sqrt{2\pi}}e^{-(y-\mu)^2/2}\frac{1}{\Gamma(\nu/2)2^{\nu/2}}e^{-z/2}z^{(\nu/2)-1}$$

over $-\infty<y<\infty$ and $0<z<\infty$. First we make the transformation from (Y,Z) to (T,U), where $T=Y/\sqrt{Z/\nu}$ and $U=\sqrt{Z}$. The inverse transformation is $Y=TU/\sqrt{\nu}$ and $Z=U^2$ over $-\infty<t<\infty$ and $0<u<\infty$. The Jacobian of the inverse transformation is

$$J = \det\begin{pmatrix} u/\sqrt{\nu} & t/\sqrt{\nu} \\ 0 & 2u\end{pmatrix} = 2u^2/\sqrt{\nu}.$$

Therefore the joint density of T and U is

$$f_{T,U}(t,u) = \frac{1}{\sqrt{2\pi}}\frac{1}{\Gamma(\nu/2)2^{\nu/2}}e^{-(tu/\sqrt{\nu}-\mu)^2/2}e^{u^2/2}u^{\nu-2}\,2u^2/\sqrt{\nu}.$$

We find the marginal density of T by integrating out U using the change of variable $x = u/\sqrt{(t^2/\nu)+1}$ as follows:

$$
\begin{aligned}
f_T(t) &= \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma(\nu/2)2^{\nu/2}} \frac{2}{\sqrt{\nu}} \int_0^\infty \exp\{-\frac{1}{2}(\frac{tu}{\sqrt{\nu}} - \mu)^2 - \frac{1}{2}u^2\} u^\nu \, du \\
&= \frac{2^{-(\nu-1)/2}}{\sqrt{\nu\pi}\Gamma(\nu/2)} \int_0^\infty \exp\{-\frac{u^2}{2}(\frac{t^2}{\nu}+1) + \frac{tu\mu}{\sqrt{\nu}} - \frac{\mu^2}{2}\} u^\nu \, du \\
&= \frac{2^{-(\nu-1)/2}}{\sqrt{\nu\pi}\Gamma(\nu/2)} e^{-\mu^2/2} \int_0^\infty \exp\{-\frac{x^2}{2} + \frac{tu\mu}{\sqrt{t^2+\nu}} u^\nu\} \, du \, (\frac{t^2}{\nu}+1)^{-(\nu+1)/2} \\
&= \frac{2^{-(\nu-1)/2}}{\sqrt{\nu\pi}\Gamma(\nu/2)} (\frac{t^2}{\nu}+1)^{-(\nu+1)/2} \exp\{\frac{-\mu^2}{2} + \frac{t^2\mu^2}{2(t^2+\nu)}\} \int_0^\infty \exp\{-\frac{1}{2}(x - \frac{t\mu}{\sqrt{t^2+\nu}})^2\} x^\nu \, dx \\
&= \frac{2^{-(\nu-1)/2}\nu^{\nu/2}}{\sqrt{\pi}\Gamma(\nu/2)} (t^2+\nu)^{-(\nu+1)/2} \exp\{-\frac{\nu\mu^2}{2(t^2+\nu)}\} \int_0^\infty \exp\{-\frac{1}{2}(x - \frac{t\mu}{\sqrt{t^2+\nu}})^2\} x^\nu \, dx.
\end{aligned}
$$

3.1.4. Since there exists an orthogonal matrix $\mathbf{P}$ such that $\mathbf{P}\mu = (\gamma, 0, \ldots, 0)^T$, we may transform the problem to $\mathbf{Z} = \mathbf{PY}$ where $X = Z_1^2 + \cdots + Z_n^2$ with $Z_1, \ldots, Z_n$ independent random variables with $Z_1 \in \mathcal{N}(\gamma, 1)$ and $Z_j \in \mathcal{N}(0,1)$ for $j = 2, \ldots, n$. We are to show that X has density (3.18).

If the result is true for $n = 1$, then it is clearly true for $n > 1$ since we just take the result for $n = 1$ and convolute it with the distribution of $Z_2^2 + \cdots + Z_n^2$, namely, the χ^2_{n-1} distribution. Then since the sum of independent chi-squares is a chi-square with the sum of the degrees of freedom, the result follows.

Therefore, suppose $n = 1$, and consider the distribution $X = Z_1^2$. This transformation is 2 to 1, with inverse transformation $Z_1 = \pm\sqrt{X}$ and Jacobian $dz_1/dx = \pm 1/(2\sqrt{x})$. Since the transformation is 2 to 1, the density of X is the sum of the two pieces,

$$
\begin{aligned}
f_X(x) &= \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{x}-\gamma)^2/2} \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{x}-\gamma)^2/2} \frac{1}{2\sqrt{x}} \\
&= \frac{1}{2\sqrt{2\pi x}} e^{-x/2-\gamma^2/2} \left[e^{\sqrt{x}\gamma} + e^{-\sqrt{x}\gamma} \right] \\
&= \frac{1}{2\sqrt{2\pi x}} e^{-x/2-\gamma^2/2} \, 2 \sum_{i \text{ even}} \frac{(\sqrt{x}\gamma)^i}{i!} \\
&= \frac{1}{\sqrt{2\pi x}} e^{-x/2-\gamma^2/2} \sum_{j=0}^\infty \frac{x^j \gamma^{2j}}{(2j)!} \\
&= \sum_{j=0}^\infty \left[\frac{e^{-\gamma^2/2}(\gamma^2/2)^j}{j!} \right] \cdot \frac{j! 2^j}{(2j)!\sqrt{2\pi}} e^{-x/2} x^{j-(1/2)}
\end{aligned}
$$

The terms in square brackets are the probabilities for $\mathcal{P}(\gamma^2/2)$. We will be finished when we show that the remaining terms are the chi-square densities, $f_{2j+1}(x)$, where

$$
f_{2j+1}(x) = \frac{1}{\Gamma(j+(1/2))2^{j+(1/2)}} e^{x/2} x^{j-(1/2)}.
$$

Thus, it is a matter of checking that the constants agree. This follows from

$$
\begin{aligned}
\Gamma(j+\tfrac{1}{2}) &= (j-\tfrac{1}{2})\Gamma(j-\tfrac{1}{2}) = \cdots = (j-\tfrac{1}{2})\cdots(\tfrac{1}{2})\Gamma(\tfrac{1}{2}) \\
&= \frac{(2j-1)(2j-3)\cdots 1}{2^j}\sqrt{\pi} = \frac{(2j)(2j-1)\cdots 1\sqrt{\pi}}{(2j)(2j-2)\cdots 2 \, 2^j} \\
&= \frac{(2j)!\sqrt{\pi}}{j! \, 2^{2j}}.
\end{aligned}
$$

3.1.5. We first derive the density of the central $\mathcal{F}_{r,n}$ distribution, and then apply (3.18). Let Y and Z be independent with $Y \in \chi^2_r$ and $Z \in \chi^2_n$. The joint density of Y and Z is

$$f_{Y,Z}(y,z) = \frac{1}{\Gamma(\frac{r}{2})2^{r/2}\Gamma(\frac{n}{2})2^{n/2}} e^{-(y/2)-(z/2)} y^{(r/2)-1} z^{(n/2)-1}$$

We want to find the density of $X = (Y/r)/(Z/n)$. We make this replacement for Y with $Y = rXZ/n$ and $dy/dx = rz/n$. Hence

$$f_{X,Z}(x,z) = \frac{1}{\Gamma(\frac{r}{2})\Gamma(\frac{n}{2})2^{(r+n)/2}} \exp\{-\frac{rxz}{2n} - \frac{z}{2}\} (\frac{rxz}{n})^{(r/2)-1} z^{(n/2)-1} \frac{rz}{n}.$$

To find the density of the central $\mathcal{F}_{r,n}$ distribution, we integrate out z, and denote the result by $g_{r,n}(x)$:

$$\begin{aligned} g_{r,n}(x) &= \frac{(r/n)^{r/2} x^{(r/2)-1}}{\Gamma(\frac{r}{2})\Gamma(\frac{n}{2})2^{(r+n)/2}} \int_0^\infty \exp\{-\frac{z}{2}(\frac{rx}{n}+1)\} z^{(r+n-2)/2}\, dz \\ &= \frac{(r/n)^{r/2} x^{(r/2)-1}}{\Gamma(\frac{r}{2})\Gamma(\frac{n}{2})2^{(r+n)/2}} \frac{\Gamma(\frac{r+n}{2})}{(\frac{rx}{n}+1)^{(r+n)/2}(\frac{1}{2})^{(r+n)/2}} \\ &= \frac{\Gamma(\frac{r+n}{2}) r^{r/2} n^{n/2}}{\Gamma(\frac{r}{2})\Gamma(\frac{n}{2})} \cdot \frac{x^{(r/2)-1}}{(rx+n)^{(r+n)/2}} \end{aligned}$$

for $x > 0$. To find the density of the noncentral $\mathcal{F}_{r,n}(\gamma^2)$, we let Y have density (3.18) with n replaced by r, and let Z be an independent χ^2_n. The joint density of Y and Z is

$$f_{Y,Z}(y,z) = \sum_{j=0}^{\infty} p_{\gamma^2/2}(j) f_{r+2j}(y) f_n(z).$$

We make the same change of variable $X = (Y/r)/(Z/n)$ for Y and integrate out Z as above to find

$$f(x|\gamma^2) = \sum_{j=0}^{\infty} p_{\gamma^2/2}(j) g_{r+2j,n}(x)$$

as the density of the noncentral $\mathcal{F}_{r,n}(\gamma^2)$. Unfortunately, this is not the same as (3.19). The correct version of (3.19) is

$$f(x|\gamma^2) = \sum_{j=0}^{\infty} p_{\gamma^2/2}(j) \frac{\Gamma(\frac{r+n}{2}+j)(r+2j)^{(r/2)+j} n^{n/2}}{\Gamma(\frac{r}{2}+j)\Gamma(\frac{n}{2})} \cdot \frac{x^{(r/2)+j-1}}{((r+2j)x+n)^{j+((r+n)/2)}}.$$

3.1.6. The density of the $\mathcal{F}_{r,n}$ distribution is

$$g_{r,n}(x) = \frac{\Gamma(\frac{r+n}{2}) r^{r/2} n^{n/2}}{\Gamma(\frac{r}{2})\Gamma(\frac{n}{2})} \cdot \frac{x^{(r/2)-1}}{(rx+n)^{(r+n)/2}}$$

for $x > 0$. The inverse of the change of variable $Y = rX/(rX+n)$ is $X = (n/r)Y/(1-Y)$ where $0 < Y < 1$, and the Jacobian is $dx/dy = (n/r)/(1-y)^2$. The density of Y is then proportional to

$$\begin{aligned} f_Y(y) &\propto (\frac{ny}{r(1-y)})^{(r/2)-1} (\frac{ny}{1-y}+n)^{-(r+n)/2} \frac{n}{r} \frac{1}{(1-y)^2} \\ &\propto y^{(r/2)-1}(1-y)^{(n/2)-1} \end{aligned}$$

for $0 < y < 1$. Thus, Y has the $\mathcal{B}e(r/2, n/2)$ distribution.

3.1.7. $X = x$ if and only if exactly x of the first $x - \alpha - 1$ balls drawn are black, and the $(x + \alpha)$th ball drawn is white. The probability of this may be computed as

$$\begin{aligned} \mathrm{P}(X = x) &= \frac{\binom{n}{x}\binom{\alpha+\beta-1}{\alpha-1}}{\binom{n+\alpha+\beta-1}{x+\alpha-1}} \cdot \frac{\beta}{n - x + \beta} \\ &= \binom{n}{x} \frac{(\alpha+\beta-1)!(x+\alpha-1)!(n+\beta-x)!}{(\alpha-1)!\beta!(n+\alpha+\beta-1)!} \cdot \frac{\beta}{n-x+\beta} \\ &= \binom{n}{x} \frac{\Gamma(\alpha+\beta)\Gamma(x+\alpha)\Gamma(n+\beta-x)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}. \end{aligned}$$

which is the mass function of $\mathcal{BB}(\alpha, \beta, n)$.