Solutions to the Exercises of Section 3.2.

3.2.1. The (i,j)-element of $\mathbf{YA}$ is $\sum_h \sum_l Y_{ih} a_{lj}$. Its expectation is $\sum_h \sum_l \mathrm{E}(Y_{ih}) a_{lj}$, which is the (i,j) element of $(\mathrm{E}\mathbf{Y})\mathbf{A}$. The (i,j)-element of $\mathbf{A^T Y^T}$ is $\sum_h \sum_l a_{hi} Y_{jl}$. Its expectation is $\sum_h \sum_l a_{hi} \mathrm{E}(Y_{jl})$, which is the (i,j) element of $\mathbf{A}^T(\mathrm{E}\mathbf{Y}^T)$.

3.2.2. The (i,j)-element of $(\mathbf{X}-\mathrm{E}\mathbf{X})(\mathbf{X}-\mathrm{E}\mathbf{X})^T$ is $(X_i - \mathrm{E}X_i)(X_j - \mathrm{E}X_j)$. Its expectation is $\mathrm{Cov}(X_i, X_j)$, which is the (i,j)-element of $\mathrm{Cov}(\mathbf{X})$.

3.2.3. If $\mathbf{A}$ is symmetric, then $\mathbf{B}^T = (\mathbf{QAQ}^T)^T = \mathbf{Q}^{TT}\mathbf{A}^T\mathbf{Q}^T = \mathbf{QAQ}^T = \mathbf{B}$, so $\mathbf{B}$ is symmetric. If $\mathbf{A}$ is nonnegative definite, then for any vector $\mathbf{b}$, $\mathbf{b}^T\mathbf{B}\mathbf{b} = \mathbf{b}^T\mathbf{QAQ}^T\mathbf{b} = (\mathbf{Q}^T\mathbf{b})^T\mathbf{A}(\mathbf{Q}^T\mathbf{b}) \ge 0$, so $\mathbf{B}$ is nonnegative definite. If $\mathbf{A}$ is positive definite and $\mathbf{Q}$ is nonsingular, then for any $\mathbf{b} \ne 0$, $\mathbf{b}^T\mathbf{B}\mathbf{b} = (\mathbf{Q}^T\mathbf{b})^T\mathbf{A}(\mathbf{Q}^T\mathbf{b}) > 0$ since $\mathbf{Q}^T\mathbf{b} \ne 0$, so $\mathbf{B}$ is positive definite..

3.2.4. Let $\mathbf{e}_j$ denote the jth unit vector. If $\mathbf{A}$ is nonnegative definite, then $a_{jj} = \mathbf{e}_j^T\mathbf{A}\mathbf{e}_j \ge 0$. If $\mathbf{A}$ is positive definite, then $a_{jj} = \mathbf{e}_j^T\mathbf{A}\mathbf{e}_j > 0$ since $\mathbf{e}_j \ne 0$.

3.2.5. If $\mathbf{A}$ is symmetric and nonnegative definite, and if $\mathbf{D} = \mathbf{PAP}^T$ is a diagonalization by an orthogonal matrix $\mathbf{P}$, then $\mathbf{A}$ is nonsingular if and only if $\mathbf{D}$ is nonsingular if and only if all elements of the diagonal of $\mathbf{D}$ are positive if and only if $\mathbf{A}$ is positive definite.

3.2.6. If $\mathbf{Y} = \mathbf{AX}+\mathbf{b}$, then $y_i = \sum_j a_{ij}x_j + \mu_i$ and $\partial y_i/\partial x_j = a_{ij}$. The Jacobian of this transformation is the determinant of the matrix with ij-component a_{ij}, namely $\det(\mathbf{A})$.

3.2.7. (a) If $\mathbf{A}$ is nonsingular, then $\mathbf{AA}^{-1} = \mathbf{I}$ and $\det\mathbf{A}\cdot\det\mathbf{A}^{-1} = \det\mathbf{I} = 1$, so that $\det\mathbf{A}^{-1} = (\det\mathbf{A})^{-1}$.

(b) If $\mathbf{A}$ is symmetric and nonnegative definite, then the square root, $\mathbf{A}^{1/2}$, satisfies $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$. Thus, $(\det\mathbf{A}^{1/2})^2 = \det\mathbf{A}$, so that $\det\mathbf{A}^{1/2} = (\det\mathbf{A})^{1/2}$.

3.2.8. We use Lemma 4. Let Σ denote the covariance matrix and note that $\det\Sigma = \sigma_1^2\sigma_2^2 - \sigma_{12}^2 = \sigma_1^2\sigma_2^2(1-\rho^2)$. The inverse of Σ is then

$$\Sigma^{-1} = \frac{1}{\det\Sigma}\begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix} = \frac{1}{1-\rho^2}\begin{pmatrix} 1/\sigma_1^2 & -\rho/\sigma_1\sigma_2 \\ -\rho/\sigma_1\sigma_2 & 1/\sigma_2^2 \end{pmatrix}.$$

Putting these values into formula (3.28), we find

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\frac{1}{\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}.$$

3.2.9. If $X \in \mathcal{N}(0,1)$ and if $Y = -X$ for $|X| \le c$ and $Y = X$ for $|X| > c$, then

$$\begin{aligned} \mathrm{P}(Y \le t) &= \mathrm{P}(Y \le t, |X| \le c) + \mathrm{P}(Y \le t, |X| > c) \\ &= \mathrm{P}(X \ge -t, |X| \le c) + \mathrm{P}(X \le t, |X| > c) \\ &= \mathrm{P}(X \le t, |X| \le c) + \mathrm{P}(X \le t, |X| > c) \qquad \text{(since } X \text{ is symmetric)} \\ &= \mathrm{P}(X \le t). \end{aligned}$$

This shows that Y has the same distribution as X i.e. $\mathcal{N}(0,1)$. The covariance of X and Y may be computed as

$$\begin{aligned} \mathrm{Cov}(X,Y) &= \mathrm{E}XY = \mathrm{E}(X^2\mathrm{I}(|X|>c)) - \mathrm{E}(X^2\mathrm{I}(|X|\le c)) = \mathrm{E}(X^2) - 2\mathrm{E}(X^2\mathrm{I}(|X|\le c)) \\ &= 1 - 2\int_{-c}^{c}\frac{x^2}{\sqrt{2\pi}}e^{-x^2/2}\,dx \end{aligned}$$

This is continuous and decreasing in c from 1 at $c=0$ to -1 at $c=\infty$. Thus, there exists a c such that X and Y are uncorrelated; yet they are dependent. Numerical methods give the value of c to be $1.538\cdots$.

3.2.10. The characteristic function of $(Y_1, \ldots, Y_n)$, given by (3.25), is

$$\phi_{\mathbf{Y}}(\mathbf{u}) = \exp\{i \sum_{j=1}^{n} u_j \mu_j - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} u_j \sigma_{jk} u_k\}$$

If $\sigma_{jk} = \sigma_{kj} = 0$ whenever $1 \le j \le r$ and $r+1 \le k \le n$, then

$$\phi_{\mathbf{Y}}(\mathbf{u}) = \exp\{i \sum_{j=1}^{n} u_j \mu_j - \frac{1}{2} \sum_{j=1}^{r} \sum_{k=1}^{r} u_j \sigma_{jk} u_k - \frac{1}{2} \sum_{j=r+1}^{n} \sum_{k=r+1}^{n} u_j \sigma_{jk} u_k\}$$

and we see the characteristic function factors into a product of a function of $u_1, \ldots, u_r$ and a function of $u_{r+1}, \ldots, u_n$. Hence the variables $Y_1, \ldots, Y_r$ are completely independent of the variables $Y_{r+1}, \ldots, Y_n$.