Solutions to the Exercises of Section 4.1.

4.1.1. Consider a decision problem with parameter space Θ, action space $\mathcal{A}$, and observations X with distribution P_θ. Suppose this problem is invariant under a group $\mathcal{G}$, and that $\mathcal{G}_1$ is a subgroup of $\mathcal{G}$. Find groups $\bar{\mathcal{G}}$ and $\tilde{\mathcal{G}}$ such that (1) for all $g \in \mathcal{G}$ and all $\theta \in \Theta$, the distribution of $g(X)$ is given by $\mathrm{P}_{\bar{g}(\theta)}$ and (2) for all $g \in \mathcal{G}$, $\theta \in \Theta$ and $a \in \mathcal{A}$, we have $L(\bar{g}(\theta), \tilde{g}(a)) = L(\theta, a)$. Since the same two statements are true with "for all $g \in \mathcal{G}$" replaced by "for all $g \in \mathcal{G}_1$", the problem is invariant under $\mathcal{G}_1$.

4.1.2. If the distribution of X is given by P_θ, then the distribution of $g_1(X)$ is given by $\mathrm{P}_{\bar{g}_1(\theta)}$ and hence, the distribution of $g_2(g_1(X)) = g_2 g_1(X)$ is given by $\mathrm{P}_{\bar{g}_2(\bar{g}_1(\theta))}$, so the family of distributions is invariant under the transformation $g_2 g_1$.

If g_1 is bimeasurable, one-to-one and onto, then the inverse transformation, g_1^{-1}, exists and is measurable. If the distribution of X is given by P_θ, then the distribution of $g_1(X)$ is given by $\mathrm{P}_{\bar{g}_1(\theta)}$. Let $Y = g_1(X)$. If the distribution of Y is given by $\mathrm{P}_{\bar{g}_1(\theta)}$, then the distribution of $g_1^{-1}(Y) = X$ is given by $\mathrm{P}_\theta = \mathrm{P}_{\bar{g}_1^{-1}\bar{g}_1\theta)}$. Hence the family of distributions is invariant under g_1^{-1}.

4.1.3. Let $\phi(x)$ be a continuous increasing function of the real line onto itself. If $X_1, \ldots, X_n$ are i.i.d. with distribution function F, and if $Y_i = \phi(X_i)$ for $i = 1, \ldots, n$, then $Y_1, \ldots, Y_n$ are i.i.d. with distribution function $G(y) = \mathrm{P}(Y \le y) = \mathrm{P}(\phi(X) \le y) = \mathrm{P}(X \le \phi^{-1}(y)) = F(\phi^{-1}(y))$. This shows that the distributions are invariant under the group $\mathcal{G}$ of transformations of the form $g_\phi(x_1, \ldots, x_n) = (\phi(x_1), \ldots, \phi(x_n))$, with $\bar{g}_\phi(F(x)) = F(\phi^{-1}(x))$. If $L(F, a) = W(F(a))$, then the loss is invariant if we can find $\tilde{g}_\phi(a)$ such that $L(F, a) = L(\bar{g}_\phi(F), \tilde{g}_\phi(a))$, or equivalently, $W(F(a)) = W(F(\phi^{-1}(\tilde{g}_\phi(a)))$ for all ϕ, F, and a. This is obviously satisfied if we take $\tilde{g}_\phi(a) = \phi(a)$. Thus the loss and hence the decision problem are invariant under $\mathcal{G}$.

4.1.4. Let $\mathbf{O}$ be an orthogonal n by n matrix. If $\mathbf{X}$ is an n-dimensional random vector with a $\mathcal{N}(\theta, \mathbf{I})$ distribution, then the distribution of $\mathbf{Y} = \mathbf{OX}$ is $\mathcal{N}(\mathbf{O}\theta, \mathbf{I})$ (since $\mathbf{OO}^T = \mathbf{I}$). This shows that the distributions are invariant under the group $\mathcal{G}$ of transformations of the form $g_{\mathbf{O}}(\mathbf{X}) = \mathbf{OX}$ with $\bar{g}_{\mathbf{O}}(\theta) = \mathbf{O}\theta$. If $L(\theta, a) = W(|\theta|, a)$, then the loss is invariant if we can find $\tilde{g}_{\mathbf{O}}(a)$ such that $L(\theta, a) = L(\bar{g}_{\mathbf{O}}(\theta), \tilde{g}_{\mathbf{O}}(a))$ or equivalently, $W(|\theta|, a) = W(|\mathbf{O}\theta|, \tilde{g}_{\mathbf{O}}(a))$ for all $\mathbf{O}$, θ, and a. This is satisfied if we take $\tilde{g}_{\mathbf{O}}(a) = a$. Thus the loss and hence the decision problem are invariant under $\mathcal{G}$.

4.1.5. Let $\mathbf{O}$ be an n by n diagonal matrix with determinant ± 1. If $\mathbf{X}$ is an n-dimensional random vector with a $\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ distribution, where $\boldsymbol{\Sigma}$ is a diagonal matrix, then the distribution of $\mathbf{Y} = \mathbf{OX}$ is $\mathcal{N}(\mathbf{0}, \mathbf{O}\boldsymbol{\Sigma}\mathbf{O})$. This shows that the distributions are invariant under the group $\mathcal{G}$ of transformations of the form $g_{\mathbf{O}}(\mathbf{X}) = \mathbf{OX}$ with $\bar{g}_{\mathbf{O}}(\boldsymbol{\Sigma}) = \mathbf{O}\boldsymbol{\Sigma}\mathbf{O}$. If $L(\boldsymbol{\Sigma}, a) = W(\det \boldsymbol{\Sigma}, a)$, then the loss is invariant if we can find $\tilde{g}_{\mathbf{O}}(a)$ such that $L(\boldsymbol{\Sigma}, a) = L(\bar{g}_{\mathbf{O}}(\boldsymbol{\Sigma}), \tilde{g}_{\mathbf{O}}(a))$ or equivalently $W(\det \boldsymbol{\Sigma}, a) = W(\det \mathbf{O}\boldsymbol{\Sigma}\mathbf{O}, \tilde{g}_{\mathbf{O}}(a))$ for all $\mathbf{O}$, $\boldsymbol{\Sigma}$, and a. This is satisfied if we take $\tilde{g}_{\mathbf{O}}(a) = a$. Thus the loss and hence the decision problem are invariant under $\mathcal{G}$.