Solutions to Exercises 5.3.1 through 5.3.3.

5.3.1. ϕ' will be as good as ϕ if it is two sided and if $\mathrm{E}_{\theta_1}\phi'(X) = .50$ and $\mathrm{E}_{\theta_2}\phi'(X) = .30$. These equations reduce to $\int_{x_1}^{x_2} dx = .5$ and $\int_{x_1}^{x_2} 2x\,dx = .7$, which integrate to $x_2 - x_1 = .5$ and $x_2^2 - x_1^2 = .7$. These equations have unique solution $x_1 = .45$ and $x_2 = .95$.

5.3.2. Since $\mathrm{E}_\theta\phi(X) = 1 - \int_{x_1}^{x_2} \theta x^{\theta-1}\,dx = 1 - x_2^\theta + x_1^\theta$ for ϕ of the form (5.33), the two equations, $\mathrm{E}_1\phi(X) = \alpha$ and $(d/d\theta)\mathrm{E}_\theta\phi(X)|_{\theta=1}$, become $x_2 - x_1 = .9$ and $x_2\log(x_2) - x_1\log(x_1) = 0$. Substituting the first equation into the second leads to $x_2\log(x_2) - (x_2 - .1)\log(x_2 - .1) = 0$, which may be solved by numerical methods. The values are $x_2 = .9196$ and $x_1 = .0196$.

5.3.3. Since $0 \le \phi(x) \le 1$ for all x, the integral $\int \phi(x)e^{\theta x}h(x)\,dx$ exists for all θ in the natural parameter space. Hence, by Lemma 3.5.3, the derivative of this integral with respect to θ may be passed beneath the integral sign. We have

$$\begin{aligned}\frac{d}{d\theta}\mathrm{E}_\theta\phi(x) &= \frac{d}{d\theta}c(\theta)\int \phi(x)e^{\theta x}h(x)\,dx \\ &= c'(\theta)\int \phi(x)e^{\theta x}h(x)\,dx + c(\theta)\int \phi(x)xe^{\theta x}h(x)\,dx \\ &= \frac{c'(\theta)}{c(\theta)}\mathrm{E}_\theta\phi(X) + \mathrm{E}_\theta X\phi(X)\end{aligned}$$

From Exercise 3.5.2, we have $c'(\theta)/c(\theta) = -\mathrm{E}_\theta X$, and the result follows.