Solutions of Exercises 5.5.1 to 5.5.3.

5.5.1. (a) If $X \in \mathcal{C}(0,1)$, and $U = 2X/(1+X^2)$, then $\mathrm{E}U = 0$ since $|U| \le 1$ and U has a symmetric distributin about 0. To evaluate $\mathrm{E}U^2$, we make the change of variable, $\theta = \arctan(x)$ with $dx = (1/\cos^2\theta)d\theta$.

$$\begin{aligned}\mathrm{E}U^2 &= \int_{-\infty}^{\infty} \left(\frac{2x}{1+x^2}\right)^2 \frac{1}{\pi(1+x^2)} dx = \frac{8}{\pi}\int_0^{\infty} \frac{x^2}{(1+x^2)^3} dx \\ &= \frac{8}{\pi}\int_0^{\pi/2} \sin^2\theta\cos^4\theta \frac{1}{\cos^2\theta} d\theta = \frac{8}{\pi}\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta\end{aligned}$$

Standard recursive methods for integrating $\cos^m\theta\sin^n\theta$ give

$$\int \sin^2\theta\cos^2\theta\, d\theta = \frac{1}{4}[\sin^3\theta\cos\theta] + \frac{1}{8}[\theta - \sin\theta\cos\theta]$$

(as may be checked by differentiating). Hence, $\mathrm{Var}(U) = \mathrm{E}U^2 = (8/\pi)(\pi/16) = 1/2$.

(b) We reject H_0 if $\sum_1^n U_i > k$, where $n = 100$. The central limit theorem gives $\sum_1^n U_i$ as approximately normal with mean zero and variance $n/2$. Hence, $\mathrm{P}(\sum_1^n U_i > k) = \mathrm{P}(\sqrt{2/n}\sum_1^n U_i > \sqrt{2/n}\,k) = .05$, gives $\sqrt{2/n}\,k$ as approximately 1.645. Hence, $k = 1.645\sqrt{50} = 11.63$.

5.5.2. Part (a) is false unless α is restricted to be less than or equal to $1 - (1/2)^n$. *Proof for* $n = 2$: Look at the contours of the likelihood ratio,

$$\varphi(x_1, x_2) = \log(f(x_1, x_2|\theta)/f(x_1, x_2|0)) = |x_1| - |x_1 - \theta| + |x_2| - |x_2 - \theta|.$$

We are to show that the sets $\{\log(f(x_1, x_2|\theta)/f(x_1, x_2|0)) > k\}$ depend on θ. If $\alpha \ge 3/4$, there is a UMP test of size α whose acceptance region is any subset of the set $\{(x_1, x_2)|x_1 < 0, x_2 < 0\}$ of probability $1-\alpha$ under H_0. However, if $\alpha < .75$, the region required to achieve size α depends on θ.

(b) The locally best test of (5.78) becomes: Reject H_0 if

$$\frac{\partial}{\partial\theta}\log f(\mathbf{x}|\theta)|_{\theta=0} = \frac{\partial}{\partial\theta} - \sum_{i=1}^n |x_i - \theta||_{\theta=0} = +\sum_{i=1}^n \mathrm{sgn}(x_i)$$

is too large. This is equivalent to rejecting H_0 if T is too large, where T is the number of positive X_i. This is the sign test. Under $H_0' : \theta = 0$, T has a binomial distribution, $\mathcal{B}(n, 1/2)$. For general θ, the distribution of T is $\mathcal{B}(n, p(\theta))$ where $p(\theta) = \mathrm{P}_\theta(X > 0) = 1 - e^{-\theta}/2$ for $\theta \ge 0$ and $p(\theta) = e^{\theta}/2$ for $\theta < 0$, so the power function can easily be computed.

(c) The test $\phi(\mathbf{X}) = \mathrm{I}(X_1 > 0)$ has power function $\beta_\phi(\theta) = \mathrm{P}_\theta(X_1 > 0) = p(\theta)$. Hence, $\beta_\phi'(\theta) = e^{-|\theta|}/2$, and $\beta_\phi''(\theta)$ does not exist at $\theta = 0$. Thus the method for finding locally best unbiased tests does not work.

5.5.3. (a) If $X_1, \ldots, X_n$ is a sample from the logistic distribution with location parameter θ, then $\log f(x_1, \ldots, x_n|\theta) = -\sum_{i=1}^n \log(2(1 + \cosh(x_i - \theta))$, and

$$\frac{\partial}{\partial\theta}\log f(x_1, \ldots, x_n)\bigg|_{\theta=0} = \sum_{i=1}^n \frac{\sinh(x_i)}{1 + \cosh(x_i)}.$$

If we let $U = \sum_{i=1}^n \sinh(X_i)/(1 + \cosh(X_i))$, then since the distribution of U is continuous, we may omit the $\gamma(x)$ term in (5.78) and conclude that the test of H_0 vs. H_1 that rejects H_0 if and only if $U > k$, is a locally best test of its size for any $k > 0$. As an aid in choosing k to achieve a preassigned size α, we note that under H_0 the distribution of $\sinh(X_i)/(1 + \cosh(X_i)$ is uniform on the interval $(-1, 1)$. This follows upon noticing that the distribution function of the logistic distribution, $\mathcal{L}(0, 1)$, may be written as $F(x) = (1/2)(1 + \sinh(x)/(1 + \cosh(x)))$ and using the fact that $F(X) \in \mathcal{U}(0, 1)$. Thus, U is the sum of n independent $\mathcal{U}(-1, 1)$'s.

(b) To find the locally best unbiased test, (5.88), we must compute

$$\frac{\partial^2}{\partial\theta^2}\log f(x_1,\ldots,x_n)\bigg|_{\theta=0} = \sum_{i=1}^{n}\frac{-(1+\cosh(x_i))\cosh(x_i)+(\sinh(x_i))^2}{(1+\cosh(x_i))^2}$$

$$= -\sum_{i=1}^{n}\frac{\cosh(x_i)+1}{(1+\cosh(x_i))^2} = -\sum_{i=1}^{n}\frac{1}{1+\cosh(x_i)}.$$

The locally best unbiased test then rejects H_0 if $V > k_1 + k_2 U$, where

$$V = -\sum_{i=1}^{n}\frac{1}{1+\cosh(x_i)} + \left(\sum_{i=1}^{n}\frac{\sinh(x_i)}{1+\cosh(x_i)}\right)^2$$

and where k_1 and k_2 are chosen so that the test is unbiased and has a preassigned size. However, the test will be unbiased if k_2 is chosen equal to zero by the argument given at the bottom of page 239, because the distribution of V is symmetric about zero when H_0 is true. Thus, the test that rejects H_0 when $V > k_1$ is a locally best unbiased test of its size.