Solutions to Exercise Set 2.

2.7.
$$\begin{aligned}\mathrm{P}(N=n) &= \mathrm{P}(X_n=1, X_{n+1}=0, X_{n+2}=0, \ldots)\\ &= \mathrm{P}(X_n=1)\mathrm{P}(X_{n+1}=0)\mathrm{P}(X_{n+2}=0)\cdots\\ &= \frac{1}{n^2}\cdot\frac{n(n+2)}{(n+1)^2}\cdot\frac{(n+1)(n+3)}{(n+2)^2}\cdots = \frac{1}{n(n+1)}\end{aligned}$$

So $\mathrm{E}(N) = \sum_1^\infty n/(n(n+1)) = \sum_1^\infty 1/(n+1) = \infty$.

3.5. (a) The characteristic function is

$$\begin{aligned}\frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha-1}e^{itx-(x/\beta)}\,dx &= \frac{1}{\Gamma(\alpha)\beta^\alpha}\int_0^\infty x^{\alpha-1}e^{-((1-it\beta)/\beta))x}\,dx\\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha}\Gamma(\alpha)\left(\frac{\beta}{1-it\beta}\right)^\alpha = \frac{1}{(1-it\beta)^\alpha}\end{aligned}$$

(b) The characteristic function of $X_n - n$ is

$$\varphi_{X_n-n}(t) = e^{-itn}(1-(it/n))^{-n^2}.$$

It is sufficient to show that $\log\varphi_{X_n-n}(t) \to -t^2/2$.

$$\begin{aligned}\log\varphi_{X_n-n}(t) &= -itn - n^2\log(1-(it/n))\\ &= -itn + n^2[(it/n) + (1/2)(it/n)^2 + O(n^{-3})]\\ &= -t^2/2 + O(n^{-1}) \to -t^2/2\end{aligned}$$

4.1. (a) The least squares estimate of λ is that value of λ that minimizes $\sum_1^n(X_i-\lambda z_i)^2$. Taking a derivative of this sum with respect to λ, setting to zero and solving gives $\hat\lambda_{LS} = \sum_1^n X_i z_i/\sum_1^n z_i^2$. The weighted least squares estimate of λ is that value of λ that minimizes $\sum_1^n(X_i-\lambda z_i)^2/z_i$. Similarly, we find $\hat\lambda_W = \sum_1^n X_i/\sum_1^n z_i$.

(b) Since $\mathrm{E}\hat\lambda_{LS} = \lambda$, it is sufficient to find for what values of the z_i we have $\mathrm{Var}(\hat\lambda_{LS}) \to 0$ as $n\to\infty$ (see Exercise 1.5). But

$$\mathrm{Var}\hat\lambda_{LS} = \mathrm{Var}\frac{\sum_1^n X_i z_i}{\sum_1^n z_i^2} = \frac{\mathrm{Var}\sum_1^n X_i z_i}{(\sum_1^n z_i^2)^2} = \frac{\sum_1^n z_i^2\mathrm{Var}X_i}{(\sum_1^n z_i^2)^2} = \frac{\lambda\sum_1^n z_i^3}{(\sum_1^n z_i^2)^2}$$

Therefore, λ_{LS} is consistent in quadratic mean if and only if $\sum_1^n z_i^3/(\sum_1^n z_i^2)^2 \to 0$ as $n\to\infty$.

(c) $\hat\lambda_W$ is also unbiased, and $\mathrm{Var}(\hat\lambda_W) = \lambda\sum_1^n z_i/(\sum_1^n z_i)^2 = \lambda/\sum_1^n z_i$. So $\hat\lambda_W$ is consistent in quadratic mean if and only if $\sum_1^n z_i \to \infty$ as $n\to\infty$.

(d) If $\hat\lambda_W$ is not consistent in quadratic mean, i.e. if $\sum_1^\infty z_i < \infty$, then both $\sum_1^\infty z_i^2 < \infty$ and $\sum_1^\infty z_i^3 < \infty$, so $\hat\lambda_{LS}$ is not consistent either. However if $z_i = 1/i$, then $\sum_1^\infty z_i = \infty$ so that $\hat\lambda_W$ is consistent in quadratic mean, but $\sum_1^\infty z_i^2 < \infty$ and $\sum_1^\infty z_i^3 < \infty$ so that $\hat\lambda_{LS}$ is not consistent.