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On Sums of Graph Games with Last Player Losing [1])

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Abstract: The purpose of this paper is to find the general class of graph games with last player losing which may be solved by an analogue of BOUTON's [1901] solution. Moreover, it can be shown that this class contains all subtraction games, as well as LASKER's [1931] nim and several other games. Games such as KAYLES and DAWSON's [1935] game with last player losing are not treated by the method of this paper and are still unsolved.

1. Introduction

A general theory of sums of graph games with last player winning was developed by SPRAGUE [1936] and GRUNDY [1939] through the use of what is now known as the SPRAGUE-GRUNDY function. An investigation and classification of many such games was undertaken by GUY and SMITH [1956]. Related work appears in HOLLADAY [1957], BERGE [1958], and SMITH [1966].

A corresponding general theory of sums of graph games with last player losing seems more difficult to develop. GRUNDY and SMITH [1956] have investigated the general problem and indicate that the function corresponding to that of SPRAGUE-GRUNDY is much more difficult to evaluate and use. This is surprising, as they point out, in view of the ease with which nim with last player losing was solved by BOUTON [1901].

It is the purpose of this paper to find the general class of graph games with last player losing which may be solved by an analogue of BOUTON's solution. This is done in Theorem 2. In the last section, it is seen that this class contains all subtraction games, as well as LASKER's nim and several other games. Games such as KAYLES and DAWSON's game with last player losing are not treated by the method of this paper and are still unsolved.

2. Graph Games

By a (*directed*) *graph* $G(X,F)$ we mean a set X of vertices and a function F mapping X into 2^X, the set of all subsets of X. For $x \in X, F(x)$ is called the set of *followers* of x. A *path* is a sequence $x_0, x_1, \ldots, x_n$ of elements of X such that

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$x_i \in F(x_{i-1})$ for $i = 1,\ldots,n$. We restrict attention to graphs that are *progressively bounded*, which means that for every $x \in X$ there is a number $C(x)$ such that if $x = x_0, x_1,\ldots,x_n$ is a path, then $n \leq C(x)$. For graphs with finite X, this means merely that there are no circuits, i.e. no paths $x_0, x_1,\ldots,x_n$ with $x_0 = x_n$. A vertex x for which $F(x)$ is empty is called *terminal*.

Given a progressively bounded graph and an initial vertex $x_0 \in X$, a two-person zero-sum game of perfect information may be played as follows. The players move alternately. The first move consists in choosing an element $x_1 \in F(x_0)$. After x_n has been chosen, the player whose turn it is to move chooses $x_{n+1} \in F(x_n)$. Play continues until a terminal vertex is reached. The rules then determine whether the last player to move wins or loses. If the rules state that moving into any terminal vertex wins (loses), we say the game is *last player winning (losing)*.

Such games are called *graph games*, and in what follows we refer to G as a game as well as a graph, and to an element of X as a position as well as a vertex.

For a progressively bounded graph $G(X,F)$, there is a function g, called the SPRAGUE-GRUNDY *function*, defined on X, such that $g(x)$ is the smallest non-negative integer not equal to $g(y)$ for any follower y of x,

$$g(x) = \min\{n \geq 0 : n \neq g(y) \quad \text{for any} \quad y \in F(x)\}. \tag{1}$$

If x is terminal, this implies that $g(x) = 0$. If the SPRAGUE-GRUNDY function of a graph is known, it is easy to describe an optimal strategy for the graph game with last player winning. Simply move, if possible, to a vertex (position) with SPRAGUE-GRUNDY function (SG-value) zero. At positions with positive SG-value, this is always possible; it is never possible at positions with SG-value zero; and all terminal positions have SG-value zero. Therefore, such a strategy is optimal.

The main value of the SPRAGUE-GRUNDY function is its use in solving sums of graph games. The *sum (disjunctive compound)* of graphs $G_1(X_1,F_1),\ldots, G_n(X_n,F_n)$ is a graph $G(X,F)$ where $X = X_1 \times X_2 \times \cdots \times X_n$, and for $x = (x_1,x_2,\ldots,x_n) \in X$, $F(x)$ is defined as

$$F(x) = \{y = (y_1,\ldots,y_n): \quad \text{for some} \quad i,\ y_i \in F_i(x_i) \quad \text{and for} \quad j \neq i,\ y_j = x_j\} \tag{2}$$

and we write $G = G_1 + \cdots + G_n$. When playing the graph game whose graph is the sum of graphs $G_1,\ldots,G_n$, a move consists in moving in exactly one of the graph games $G_1,\ldots,G_n$. The reason the SPRAGUE-GRUNDY function is useful in solving sums of graph games is that there is a very simple operation for obtaining the SPRAGUE-GRUNDY function for a sum of graphs from the SPRAGUE-GRUNDY functions of the component graphs. This operation is binary addition without carry, known as *nim-sum*. If x and y are nonnegative integers with binary expansions $x = \sum_0^m x_i 2^i$ and $y = \sum_0^m y_i 2^i$ for some m, where each x_i is zero or one, then the nim-sum of x and y is z, written $x \hat{+} y = z$, where

$z = \sum_{0}^{m} z_i 2^i$ and $z_i = x_i + y_i \bmod 2$ and $z_i = 0$ or 1. Nim sum is associative and commutative since addition mod 2 is.

The following basic theorem is due to SPRAGUE [1936] and GRUNDY [1939].

Theorem 1:

If g_i is the SPRAGUE-GRUNDY *function of G_i $i = 1,\ldots,n$, then $G = G_1 + \cdots + G_n$ has* SPRAGUE-GRUNDY *function $g(x_1,\ldots,x_n) = g_1(x_1) \hat{+} \cdots \hat{+} g_n(x_n)$.*

3. Bouton's Solutions for Nim

The game of nim, originally solved by BOUTON [1901], is played as follows. There are n piles of counters with x_i counters in the i^{th} pile, $i = 1,2,\ldots,n$. Two players alternate moving, a move consisting of removing as many counters as desired, but at least one, from any one pile. When all the counters have been removed, the game ends and the last player to move wins.

BOUTON calls a position $(x_1,\ldots,x_n)$ *safe* if it is such that the player whose turn it is to move will lose if his opponent plays optimally. Otherwise, a position is called *unsafe*. From an unsafe position, the player to move can force a win by moving to a safe position, and from thence to the end of the game always moving to a safe position. BOUTON showed that the safe positions at nim are those for which the nim-sum of the number of counters in the various piles is zero. This result can easily be seen from Theorem 1. For one-pile nim (a trivial game), the SPRAGUE-GRUNDY function of equation (1) is $g(x) = x$, where x denotes not only the number x but also the position with x counters in the pile. Hence, the SPRAGUE-GRUNDY function for n pile nim is $g(x_1,\ldots,x_n) = x_1 \hat{+} \cdots \hat{+} x_n$.

BOUTON also finds the safe positions for nim with last player losing. Let

$$B_0 = \{(x_1,\ldots,x_n) : x_1 \hat{+} \cdots \hat{+} x_n = 0 \quad \text{and for some} \quad i,\ x_i \geq 2\}$$

and let

$$B_1 = \{(x_1,\ldots,x_n) : x_1 \hat{+} \cdots \hat{+} x_n = 1 \quad \text{and for all} \quad i, x_i \leq 1\}.$$

Then the set $B = B_0 \cup B_1$ is the set of safe position for nim with the last player losing. BOUTON's method of proof is to check the three conditions: *i*° every follower of a position in B is not in B, *ii*° every position not in B has a follower in B and *iii*° no terminal position is in B.

In the next section, conditions on the component graphs are presented which allow one to show by the same method that the safe positions consist of the analogue of BOUTON's positions. The structure of the graph is first generalized to allow one to treat games in which splitting of one pile of counters into two or more piles is allowed. These considerations, which are more or less trivial for games with the last player winning, complexify the analysis but enlarge the scope of the applicability of the theory.

4. The Main Theorem

We say a graph $G(X,F)$ may be *represented as a sum at a vertex* $x_0 \in X$, if the graph $G_0(X_0,F_0)$ with $X_0 = \{x \in X$: there is a path from x_0 to $x\}$ and $F_0(x) = F(x)$ for $x \in X_0$, is representable as a sum $G_1(X_1,F_1) + \cdots + G_n(X_n,F_n)$. In such a case, we will write $x_0 = (x_1,\ldots,x_n)$ with $x_i \in X_i$ and speak of the x_i as components of the position x_0. It may happen in one of the component games, say G_n, that we arrive at a vertex $x_n \in X_n$ at which G_n itself may be represented as a sum $G_n = G_{n1} + \cdots + G_{nm}$. Then it turns out advantageous to represent the game G at such a vertex as the sum $G_1 + \cdots + G_{n-1} + G_{n1} + \cdots + G_{nm}$.

Therefore, we speak of the vertices of a graph as having components, each component being the vertex of one of the component graphs. The component itself may have followers, some of which may consist of more than one component. We assume that the structure of the graph is given; that is, we assume it is decided which of various component graphs shall be represented as a sum at which vertices. Theorem 1 holds at all vertices. The SG-value of a position is the nim-sum of the SG-values of the components.

We list the assumptions on the structure of the graph to be used in the following theorem.

A1. Every non-terminal component with SG-value zero has a follower with SG-value one.

A2. If x is a component with SG-value zero and if y is a follower of x with SG-value one, then every component of y has SG-value zero or one.

A3. If x is a component with SG-value one and if y is a follower of x with SG-value zero, then every component of y has SG-value zero or one.

A4. If x is a component with SG-value at least two, then either there exists a follower of x with SG-value one all of whose components have SG-values zero or one, or there exists a follower of x with SG-value zero having at least one component of SG-value at least two.

A5. If x is a component with SG-value at least two, then either there exists a follower of x with SG-value zero all of whose components have SG-values zero or one, or there exists a follower of x with SG-value one having at least one component of SG-value at least two.

The analogue of BOUTON's positions for graphs of the above structure may be described as follows.

Let B_0 be the set of positions of SG-value zero at least one of whose components has SG-value two or greater.

Let B_1 be the set of positions of SG-value one all of whose components have SG-value zero or one.

Finally, let $B = B_0 \cup B_1$.

Theorem 2:

If conditions A_1 through A_5 are satisfied, then the set B is the set of safe positions for the game with last player losing.

Proof:

Clearly, no terminal position is in B. The theorem will be proved by verifying $i°$ every follower of a position in B is not in B, and $ii°$ every nonterminal position not in B has a follower in B.

$i°$. Suppose $x \in B_0$; then every follower of x has SG-value different from zero and at least one component with SG-value two or greater, and hence is not in B. Suppose $x \in B_1$. If every component of a follower of x has SG-value zero or one, then the SG-value of the follower is zero and hence is not in B. If a follower of x has a component of SG-value two or greater, then the SG-value of the follower cannot be zero from conditions *A2* and *A3*. Hence no follower of x is in B.

$ii°$. Suppose $x \notin B$ and has some component with SG-value at least two. Then there is a follower of x with SG-value zero. If some component of the follower has SG-value at least two, the follower is in B. If all components of the follower have SG-values zero or one, then the follower has been obtained by changing the only component c of x that has SG-value at least two. If there are an even number of components of x with SG-value one, then by *A4* there is a follower of x that is in B and obtained by changing c. If there are an odd number of components of x with SG-value one, then by *A5* there is a follower of x that is in B and obtained by changing c.

If x is a nonterminal position not in B such that all components of x have SG-values zero or one, then the SG-value of x is zero, and from conditions *A1*, *A2*, and *A3* there is a follower of x that is in $B_1 \subset B$. ■

Conditions *A1* through *A5* are very nearly necessary as well as sufficient for B to be the set of safe positions. If condition *A1* is not satisfied, there is a nonterminal component with SG-value zero all of whose followers have SG-value at least two. The position all of whose components are terminal except this one is a position not in B all of whose followers are not in B. Therefore *A1* is a necessary condition. A similar analysis shows that conditions *A3* and *A4* are necessary too.

Condition *A5* is not quite necessary as the following example shows. The first move consists of choosing between playing nim with two piles of two matches each, and playing nim with one pile of one match. Here, condition *A5* is not satisfied at the initial position, and yet B is indeed the set of safe positions. However, if such a game is a component of a position with at least one other nonterminal component, then B is not the set of safe positions. For example, the initial position of the sum of the above game and a game of one pile, one match nim is safe though not in B. A similar example shows that *A2* is not quite necessary.

If every follower of each component consists of only one component, as is the case when G is a simple sum $G_1 + \cdots + G_n$, then conditions *A2, A3, A4* and *A5* are trivially satisfied. This observation leads to the following corollary. In this case, condition *A1* can be imposed on the component games, G_i for $i = 1,\ldots,n$. If g_i represents the SPRAGUE-GRUNDY function for G_i, condition *A1* means that if x is nonterminal and $g_i(x) = 0$, then there is a follower y of x in G_i such that $g_i(y) = 1$.

Corollary:

B is the set of safe positions for the game $G_1 + \cdots + G_n$ with last player losing, if and only if, for $i = 1,\ldots,n$ G_i satisfies condition A1.

5. Applications

1. Sums of subtraction games. Let S be a nonempty subset of the positive integers. *A subtraction game with subtraction set S*, denoted by G_S, is the game played with a single pile of counters in which at each turn a player is allowed to remove s counters from the pile provided $s \in S$ and there are at least s counters in the pile. One pile nim is G_S when S is the set of all positive integers.

Let G_S be a subtraction game with subtraction set S and let g_S be the SPRAGUE-GRUNDY function. Then condition *A1* becomes the following: If x is not terminal and $g_S(x) = 0$, then there is a $y \geq 0$ such that $x - y \in S$ and $g_S(y) = 1$.

Theorem 3:

Subtraction games satisfy condition A1. The proof of this theorem is based on the following lemma that is of separate interest.

Lemma 1.

Let k be the smallest element of S. Then $g_S(x) = 0$ implies $g_S(x + k) = 1$. Conversely, $g_S(x) = 1$ implies $g_S(x - k) = 0$.

Proof:

Since $k \in S$, $g_S(x) = 0$ implies $g_S(x + k) \neq 0$. Assume the conclusion is false and find the smallest x such that $g_S(x) = 0$ and $g_S(x + k) \geq 2$. Since $g_S(x + k) > 1$, there is an $s \in S$ such that $g_S(x + k - s) = 1$. Then $x - s \geq 0$ since k is the smallest element of S. Furthermore, $g_S(x) = 0$ implies $g_S(x - s) > 0$. Thus, there exists an $s' \in S$ such that $g_S(x - s - s') = 0$. This, together with $g_S(x - s + k) = 1$ entails $g_S(x - s - s' + k) \geq 2$. Thus, $y = x - s - s' < x$ also satisfies $g_S(y) = 0$ and $g_S(y + k) \geq 2$ contradicting the choice of x as the smallest such integer.

Conversely, if $g_S(x) = 1$ and $g_S(x - k) \neq 0$, then there is an $s \in S$ such that $g_S(x - k - s) = 0$. From the first part of the theorem, this implies $g_S(x - s) = 1$. This contradicts $g_S(x) = 1$. ■

Proof of Theorem 3:

Given any nonterminal x such that $g_S(x) = 0$, one has $g_S(x - k) \neq 0$, where k is the smallest element of S. This implies that there is an $s \in S$ such that $g_S(x - k - s) = 0$. From the lemma, $g_S(x - s) = 1$. ∎

We see that one-pile subtraction games with last player losing are easy to play. Always move, if possible, to a position with SG-value one. Sums of such games are correspondingly simple. Always move to a position in B.

2. LASKER's *nim*. A modification of the rules of nim suggested by E. LASKER [1931] allows a player at his turn the options of removing any positive number of counters from a pile, and of splitting a pile into two nonempty piles without removing any counters. The SPRAGUE-GRUNDY function of LASKER's nim, denoted by g_L, has been evaluated by SPRAGUE [1936] and found to be

$$g_L(0) = 0$$

and for $x > 0$

$$g_L(x) = \begin{cases} x & \text{if } x = 1 \text{ or } 2 \bmod 4 \\ x + 1 & \text{if } x = 3 \bmod 4 \\ x - 1 & \text{if } x = 0 \bmod 4 \end{cases}$$

The only component of SG-value zero is the empty pile which is terminal. Therfore, *A1* and *A2* are automatically satisfied. The only component of SG-value one is the pile with one counter, and its only follower is terminal. Therefore *A3* is satisfied. All other components have the empty pile and the pile of one counter as followers. Therefore, *A4* and *A5* are satisfied. Hence for LASKER's nim with last player losing, B is the set of safe positions.

In fact, the above analysis holds for any game such that $i°$ every nonterminal component has a terminal follower, and $ii°$ every nonterminal component not all of whose followers are terminal has as a follower a position all of whose followers are terminal. In particular, B is the set of safe positions for SPRAGUE's variants of LASKER's nim given in SPRAGUE [1936]. One variant allows splitting into any even number of nonempty piles. This variant has the same SPRAGUE-GRUNDY function as LASKER's nim. Another variant allows splitting into any number of nonempty piles. This variant has SPRAGUE-GRUNDY function.
$g(0) = 0$, $g(1) = 1$, $g(2) = 2$, $g(3) = 4$, $g(4) = 3$, $g(5) = 5$, $g(6) = 6$, and for $n \geq 1$, $g(5n + 2) = 8n$, $g(5n + 3) = 8n + 2$, $g(5n + 4) = 8n + 1$, $g(5n + 5) = 8n + 5$, $g(5n + 6) = 8n + 7$.

3. *Split or take one to k.* Consider the game in which a player may, at his turn, split any one pile of counters into two nonempty piles, or remove from any one pile any positive number of counters not exceeding k. For k even, the SPRAGUE-GRUNDY function is eventually periodic of period $k + 2$. In terms of g_L for LASKER's nim it may be written

$$g(x) = g_L(x) \quad \text{for} \quad x \leq k,\ g(k + 1) = 0,\ g(k + 2) = k + 1$$

and

$$g(x) \dot{=} g(x - (k + 2)) \quad \text{for} \quad n > k + 2 .$$

For k odd, it is eventually periodic of period $k + 1$. For $k = 1 \bmod 4$,

$$g(x) = \begin{cases} g_L(x) & \text{for} \quad x \leq k + 1 \\ g_L(x - 1) \mod k + 1 & \text{for} \quad x > k + 1 \end{cases}$$

with $0 \leq g(x) \leq k$. For $k = 3 \bmod 4$,

$$g(x) = \begin{cases} g_L(x) & \text{for} \quad x \leq k + 1 \\ g_L(x + 1) - 2 \mod k + 1 & \text{for} \quad x > k + 1 \end{cases}$$

with $0 \leq g(x) \leq k$. For example, the SG-series for $k = 4$ is $0\,\dot{1}\,2\,4\,3\,0\,\dot{5}$ (the sequence between the dots is to be repreated indefinitely), for $k = 5$ it is $0\,1\,2\,4\,3\,5\,\dot{6}\,0\,2\,1\,3\,\dot{4}$, for $k = 6$ it is $0\,\dot{1}\,2\,4\,3\,5\,6\,0\,\dot{7}$ and for $k = 7$, it $0\,1\,2\,4\,3\,5\,6\,8\,\dot{7}\,0\,2\,1\,3\,4\,6\,\dot{5}$.

For k even, it is not difficult to check that conditions *A1* through *A5* are satisfied, so that from Theorem 2 the set B is the set of safe positions in the game with last player losing.

For k odd, the game does not satisfy condition *A1* at $x = k + 2$. Hence, B is not the set of safe positions.

4. The game ·770. An interesting game solved by GUY and SMITH [1956] and denoted by ·770 in their notation is played with the following rules. At each turn, a player may remove from any one pile any number of counters not divisible by three; he may also, if he likes, split that pile into two non-empty piles. The SPRAGUE-GRUNDY function for this game is eventually periodic of period three. The SG-series starts out 0 1 2 3 1 4 3 6 4 3 6 7... and does not settle down to periodicity until the 160$^{\text{th}}$ term, there being 126 exceptional values.

Yet one can show fairly easily that conditions *A1* through *A5* are satisfied. From an examination of the SG-series given in GUY and SMITH [1956], one sees that there are no nonterminal zero SG-values; hence *A1* and *A2* are satisfied. There are only two components with SG-value one; namely 1 and 4. The followers of SG-value zero are 0 and (1,1); hence *A3* is satisfied. For x not divisible by 3, the component x has 0 as a follower, while an x divisible by 3 has (1,1) as a follower; hence *A5* is satisfied. If x is not equal to one mod 3, x has 1 as a follower. If x is equal to one mod 3 and $x \geq 7$, x has (3,3) as a follower. Thus *A4* is satisfied as well and the BOUTON positions are the safe positions.

5. Other games. Conditions *A1*, *A2* and *A3* are relatively strong and conditions *A4* and *A5* relatively weak. Therefore, if an SG-series has few zeros and ones we suspect it is more likely to satisfy the conditions of Theorem 2. This is borne out statistically if we look at tables 2 and 3 of GUY and SMITH [1956]. Of the games listed in table 2 in which the SG-series have many zeros and ones, only about

60% satisfy conditions *A1* through *A5*. On the other hand, the games listed in table 3 have only a few zeros and ones that appear only at the very beginning of the SG-series. Of these 77 games, only one does not satisfy conditions *A1* through *A5*, that one being $\cdot 7\dot{7}\dot{0}$, which does not satisfy *A5*.

It should be noted that the SG-series alone does not determine whether a game satisfies *A1* through *A5*. For example, the game ·333 (the subtraction game with $S = \{1,2,3\}$) and the game ·73 (take 1, or take 2, or take 1 and split) have the same SG-series 0 1 2 3 0 1 2 3... and so are equivalent as games with the last player winning. Yet the former satisfies the conditions from Theorem 3, and so has a simple solution for the last player losing. The latter does not satisfy conditions *A1* and *A3*, and its solution is not so simple.

In spite of the apparent success of Theorem 2, it should be pointed out that most of the more interesting games do not satisfy the conditions of Theorem 2. The well-studied game of KAYLES in which a player at his turn may take 1, or take 2, or take 1 and split, or take 2 and split, has SG-series eventually periodic of period 12 that commences 0 1 2 3 1 4.... This does not satisfy condition *A5* for a component with 5 counters. DAWSON's game, originally suggested and partially analysed as a game with last player losing by DAWSON [1935], in which a player at his turn may take 2, or take 3, or take 3 and split, or take 1 provided it is a whole pile, has an SG-series eventually periodic of period 34 that commences 0 1 1 2 0 3 1 1 0 3.... This does not satisfy condition *A5* for a component with 9 counters. These two games and the others mentioned in GUY and SMITH that do not satisfy the conditions *A1* through *A5*, and the game, split or take one to k, for k odd, seem very difficult to treat as last player losing games and are still unsolved.

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