Models for the Game of Liar's Dice

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Abstract: An explicit multimove game of competition where a player must occasionally lie and the other must detect the lie is solved.

1. Introduction: Models of competition between two decision makers, where one decision maker must occasionally lie and the other must attempt to detect the lie, were described in a nonsequential setting in T. S. Ferguson (1970). Here, several analogous sequential models directly related to liar's dice are treated, in which the decision makers alternately exchange roles. The bigger the successful lie at one stage, the more difficult it is for the opponent at the next stage.

The 1970 model is as follows. Player I observes x chosen from a uniform distribution on $(0, 1)$, and claims that he observes $y \in (0, 1)$ where $y \geq x$. Player II, not knowing x, is informed of y and must accept or challenge I's claim. If II challenges, I wins 1 from II if $y = x$ and wins nothing otherwise. If II accepts, then I wins $b(x, y)$ from II, where $b(x, y)$ is a known function of x and y. The value and optimal strategies for the players were found for this game under certain monotonicity and differentiability conditions on $b(x, y)$. As examples, the functions $b(x, y) = y$, and $b(x, y) = (y - x)/(1 - x)$ were solved explicitly.

No sequential models were treated in Ferguson (1970), although hope was expressed that the results would have application to such models. We quote: "Use of a general $b(x, y)$ allows treatment of situations wherein the basic game is replayed with the roles of the players reversed and future x and y dependent upon past x and y." This may be true in principle, but difficult to carry out in practice.

In this paper, we treat several sequential models directly, all models of the game of liar's dice as described in Bell (1969), Frey (1975) or Scarne (1980). In section 2,

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a one die version of the game is solved. This is a finite game with at most 6 rounds of play, but with at most 2 rounds under optimal play. In section 3, a continuous version is treated which models liar's dice in which all dice must be rerolled in each round. Although actually an infinite game, under optimal play there is at most 2 rounds. This makes it the game $b(x, y) = y$ in the Ferguson (1970) model, and it has the same solution as given there. In section 4, we treat a version of the game most closely resembling the game of liar's dice as it is played, the seven category game. In section 5, we treat a continuous version modeling liar's dice in which a player may choose which dice to reroll. We call this the continuous improvable case. It is a true stochastic game in the sense of Shapley (1953) (but with continuous state space and some zero stop probabilities) with unbounded length of play under optimal strategies. In section 6, we allow a player to hide from his opponent which of the dice he rerolls. We note that it is advantageous occasionally to try deliberately to get a poor combination of the dice.

2. The Liar's Die Game: As an introductory example for this class of games, we consider the game in which there is just one die. Player I rolls the die and observes the outcome, a random integer $X(1)$ taking the values from 1 to 6 with equal probabilities. Then, based on $X(1)$, player I chooses an integer $y(1)$ between 1 and 6 inclusive and makes the claim that the $X(1)$ is (at least) $y(1)$. Then, player II announces whether he doubts or believes I's claim. If II doubts I's claim, then II wins if $X(1) < y(1)$, and I wins otherwise. If II believes, then the game is played over with the roles of the players reversed. Player II rolls the die but this time II must claim a number that is higher than the number previously claimed by his opponent. Thus, II observes a random integer $X(2)$ between 1 and 6, chooses a number $y(2) > y(1)$, and claims that $X(2) \geq y(2)$. Play proceeds similarly in subsequent rounds. If either player claims 6, then the opponent may as well doubt the claim, since that gives the only possible way to win.

We take the payoff to be 1 if player I wins and 0 if player II wins; this is so that the value of the game will represent player I's probability of win under optimal play by both players. The rules allow the game to be repeated at most six rounds, but under optimal play there are at most two rounds as the following theorem indicates.

Theorem 1: (i) The value of the liar's die game is $41/60$.

(ii) An optimal strategy for player I is as follows. If $X(1) \geq 3$, then claim $y(1) = X(1)$. If $X(1) = 2$, then claim $y(1) = 2$ with probability $3/10$, and claim $y(1) = 3$ otherwise. If $X(1) = 1$, then claim $y(1) = 3$ with probability $3/10$, $y(1) = 4$ with probability $5/10$, and $y(1) = 5$ with probability $2/10$. On the second round (if any), doubt any claim by II.

(iii) An optimal strategy for player II is as follows. If $y(1) = y$, then doubt I's claim with probability $p(y)$, where $p(1) = 0$, $p(2) = 0$, $p(3) = 1/3$, $p(4) = 1/2$, $p(5) = 3/5$, $p(6) = 1$. On the second round (if any), claim the minimum of $X(2)$ and $y(1) + 1$.

Proof: Suppose I uses the indicated strategy. What is II's best reply? If II hears $y(1) = y$, he must doubt if $y = 6$; if $y = 5$, then the probability that $X(1) = 5$ is $P(X(1) = 5|y(1) = 5) = P(X(1) = 5, y(1) = 5)/P(y(1) = 5) = (1/6)/((1/6) + (1/6)(2/10)) = 5/6$; if $y = 4$, then $P(X(1) = 4|y(1) = 4) = 2/3$; if $y = 3$, then $P(X(1) = 3|y(1) = 3) = (1/6)/((1/6) + (1/6)(3/10) + (1/6)(7/10)) = 1/2$.

If II doubts when $y(1) = 5$, he wins with probability $1/6$. If II believes, he wins if and only if he rolls a six (since I is going to doubt whatever he does), so here too he wins with probability $1/6$. In other words, II is indifferent whether to doubt or believe. Similarly, if $y(1) = 4$, II wins with probability $1/3$, whether he doubts or believes, and if $y(1) = 3$, II wins with probability $1/2$, whether he doubts or believes. If $y(2) = 2$, clearly II should believe since I is telling the truth, and then since he knows I is going to doubt his claim, which must be at least 3, he should tell the truth if he can and so will win with probability $2/3$. All in all, II may as well doubt any $y(1) > 2$ and believe $y(1) = 2$, giving I a probability of win $= 2/3 + (1/6)(3/10)(1/3) = 41/60$, as claimed.

Now suppose II uses the indicated strategy. What is I's best counter strategy? Suppose I sees $X(1) = 1$ or 2; if he claims 3, 4, or 5, he wins with probability $(2/3)(1/2) = 1/3, (1/2)(2/3) = 1/3$, or $(2/5)(5/6) = 1/3$ resp.; if he claims 6, he loses with probability 1; if he claims 2 (1 is worse), he wins with probability $1/3$ also. If I sees $X(1) = 3, 4, 5$, or 6, his best policy is honesty, and he wins with probability $(1/3) + (2/3)(1/2) = 2/3, (1/2) + (1/2)(2/3) = 5/6, (3/5) + (2/5)(5/6) = 14/15$, or 1 resp. I's overall probability of winning is therefore $(1/6)(1/3 + 1/3 + 2/3 + 5/6 + 14/15 + 1) = 41/60$, as claimed. ◇

3. The Continuous Liar's Dice Game: Player I observes the outcome of a random variable $X(1)$ having a continuous distribution function, $F(x)$. He then chooses $y(1)$ and claims that $X(1) \geq y(1)$. Player II then must either challenge or accept II's claim. If he challenges, Player I wins if and only if he was telling the truth. If II accepts, then the game is played again with the roles of the players reversed; II observes $X(2)$ from $F(x)$, independent of $X(1)$, and claims $X(2) \geq y(2)$, but this time $y(2)$ must be larger than $y(1)$. The game may be repeated indefinitely, with the players reversing roles, and the new call always greater than the previous call.

A closely related game was analyzed in [3] in which the rules have been altered so that if the second round is reached, player I wins if and only if $X(2) \leq y(1)$. One may think of this as the game in which if the second round is reached, I must challenge II's call. It turns out that this is an optimal strategy for I on the second round so that the value found in [3] is also the value of the game here, namely

$1 - 1/e$. The optimal strategies found there are also optimal first round strategies here, but we give a different (nonrandomized) optimal first round strategy for I in the theorem below.

Since the game depends only on the relative sizes of the $X(n)$ and not on their actual values, we may and do assume without loss of generality that $F(x)$ is the uniform distribution on the interval $[0, 1]$.

Theorem 2: The value of the Continuous Liar's Dice Game is $1 - 1/e$. Optimal for I is the strategy: if $X(1) > 1/e$, then call $y(1) = X(1)$; if $X(1) \leq 1/e$, then call $y(1)$ where $y(1) - 1 - ln(y(1)) = X(1)$; on the second round, challenge any call of II. Optimal for II is the strategy: accept any $y(1) < 1/e$; accept a call of $y(1) > 1/e$ with probability $(1/e)/y(1)$; on the second round, use I's first round strategy except that if it turns out $y(2) < y(1)$, put $y(2)$ anywhere above $y(1)$; on the third round, challenge any call of I.

Proof: Suppose I uses the indicated strategy, and II hears a call of $y(1) = y$. What is the probability I is telling the truth? Let us compute the density of the random quantity $Y = y(1)$ given by I's strategy. This is a change of variable problem with a 2-to-1 transformation. The density of Y is therefore the sum of the two pieces corresponding to $0 < x < 1/e$ and $1/e < x < 1$. The contribution of the first piece is $|dx/dy|$ where $x = y - 1 - ln(y)$, namely, $|1 - 1/y| = 1/y - 1$. The contribution of the second is 1, since the transformation is $x = y$. On hearing $y(1) = y$, the probability that I is telling the truth is therefore the probability that the source is the second contribution, namely, $1/(1 + (1/y - 1)) = y$. If II challenges, then I wins with probability y, but if II accepts, then since I is going to challenge anything II claims, I will win if and only if $X(2) < y$, which also occurs with probability y. Thus it makes no difference what II does against I's strategy; II may as well challenge everything. The probability I wins is then $1 - 1/e$.

Now suppose II uses his indicated strategy, and I observes a value $X(1) = x$. What should I claim? If I claims $y > 1/e$, $y > x$, then I wins iff II accepts and then loses; against II's second round strategy, the best I can do is to challenge any claim of II, so I wins with probability at most $((1/e)/y)y = 1/e$. If $x < 1/e$ and I claims $y > x$ and $y < 1/e$, then II accepts and since on the next round he uses I's original strategy, II wins with probability at least $1 - 1/e$; so again I wins with probability at most $1/e$. If $x < 1/e$ and I calls $y = x$, I wins with probability at most $1/e$. If $x > 1/e$ and I calls $y = x$, then I wins with probability at most $(1-p(y)) + p(y)y$ where $p(y) = (1/e)/y$. Since this is greater than $1/e$, I should tell the truth if $x > 1/e$, and it is immaterial whether I lies or tells the truth if $x \leq 1/e$. This strategy holds I's probability of win down to

$$\int_0^{1/e} (1/e)dx + \int_{1/e}^{1} (1 - p(x) + p(x)x)dx = 1 - 1/e.$$

◇

4. Seven-Category Liar's Dice: In this section, we consider liar's dice as it is usually played, allowing a player who accepts a call to leave some of the dice on the table and to roll only those dice that he wishes. The dice not rolled are left in view of his opponent. We consider the realistic case in which there are five dice and to avoid cumbersome details, we restrict attention to seven-category liar's dice, in which if a player accepts a claim by an opponent he must then call a higher category. The seven categories in decreasing order are taken to be 5-of-a-kind, 4-of-a-kind, full house, 3-of-a-kind, two pairs, one pair, and 5 different. Note that a straight is considered as a low hand. Straights are relatively infrequent in games in which it is allowed and it does not pay to try to draw to it. The probabilities of these seven categories are computed in Epstein (1967) and listed in the following table.

Category	# rolls	probability
1. 5 different	720	.0926
2. One pair	3600	.4629
3. Two pairs	1800	.2315
4. 3-of-a-kind	1200	.1543
5. Full house	300	.0386
6. 4-of-a-kind	150	.0193
7. 5-of-a-kind	6	.0008
Total	7776	1.0000

The rules of seven-category liar's dice are as follows. Player I rolls five dice and claims the dice resulted in one of the seven categories above. Player II, not knowing the outcome of the dice must accept or challenge I's claim. If II challenges I's claim, then I wins iff the dice are in the category claimed by I. If II accepts I's claim, the dice are shown to II and the game is repeated with the roles of the players reversed, but this time II may leave some of the dice as they are and roll the rest. The dice not rolled are shown to the opponent and only the new dice are hidden. Now II must claim a higher category than was claimed previously, and it is I's turn to accept or challenge II. Play continues in this fashion until one of the players challenges the other. If a player ever claims the 5-of-a-kind category, it may be assumed that the opponent will challenge, since there is no higher category to call.

This game is less in player I's favor than the game in which all five dice must be rerolled each time a player challenges. It turns out that using optimal strategies, the game never lasts more than three rounds. The critical case occurs when on the first round I truthfully claims category 2 and II accepts the claim. We denote the game continuing from this point by $G(2,2)$ and discuss its solution first.

Suppose then that the last roll resulted in one pair. It is clear that the roller should leave the pair down and roll the remaining three dice. Using such a strategy, the probabilities of improvement to other categories are:

Category	# rolls	probability
2. One pair	60	.2778
3. Two pairs	60	.2778
4. 3-of-a-kind	60	.2778
5. Full house	20	.0926
6. 4-of-a-kind	15	.0694
7. 5-of-a-kind	1	.0046
Total	216	1.0000

The roller should be truthful if he improves, that is, if he rolls one of the categories 3 through 7. If he does not improve, he should claim category y with probability $Q(y)$, where

$$\begin{aligned} Q(6) &= 9/215 = .0419 \\ Q(5) &= 11/100 = .1100 \\ Q(4) &= 8/15 = .5333 \\ Q(3) &= 1 - \text{ the above } = .3148 \end{aligned} \tag{4.1}$$

The main property of this strategy is that the second player is indifferent as to what he should do if he hears a 4, 5 or 6. For example, suppose he hears a 6; then the probability the roller has 4-of-a-kind is $(15/216)/(15/216 + (60/216)(9/215)) = 215/251$. If he challenges, he wins with probability $36/251$; if he accepts, he wins only if he can roll 5-of-a-kind (since his opponent will surely challenge), which happens with probability $1/6$ if he has 4-of-a-kind and with probability $1/216$ if he has a pair, and so with overall probability $(215/251)(1/6) + (36/251)(1/216) = 36/251$. Thus he is indifferent if he hears a 6, and similarly if he hears a 4 or a 5. If he hears a 3, he should accept and, since his opponent will challenge any claim he makes, should lose with probability $120/216$. His overall probability of losing may be computed assuming he challenges 4,5,6 and 7 to be

$$(1 + 15 + 20 + 60)/216 + (120/216)(60/216)(1 + .3148) = .6473\ldots \tag{4.2}$$

The roller can win with this probability at least. To see that this is the value of $G(2,2)$, we find the optimal strategy of the second player. The second player should challenge a claim of y with probability $p(y)$, where $p(2) = p(3) = 0$, $p(7) = 1$, and $p(4)$, $p(5)$ and $p(6)$ are chosen to make the roller indifferent concerning what to call if he rolls a 2:

$$\begin{aligned} (1 - p(4))(180/216) &= (1 - p(5))(200/216) \\ &= (1 - p(6))(215/216) = 120/216 \end{aligned} \tag{4.3}$$

which gives $p(4) = 1/3$, $p(5) = 2/5$, and $p(6) = 19/43$. Since this guarantees the second player that he loses with probability no more than (4.2), this must be the value of the $G(2,2)$.

Theorem 3: The value of seven-category liar's dice is .55919... . I's optimal strategy is as follows. On the first round, if I rolls category 3,4,5,6 or 7, he claims the truth; if I rolls category 1, he claims category 2; if I rolls category 2, he claims category x with probability $P(x)$, where

$$\begin{aligned} P(6) &= 3/430 = .007 \\ P(5) &= 11/400 = .027 \\ P(4) &= 8/45 = .178 \\ P(3) &= 2/5 = .400 \\ P(2) = 1 - &\text{ the above } = .388 \end{aligned}$$

On the second round, I should challenge any claim by II unless I truthfully claimed category 2 in the first round, in which case he should challenge a claim of y by II with the probabilities $p(y)$, associated with (4.3).

II's optimal strategy is as follows. On the first round, II should challenge a claim of y by I with probability $q(y)$ where $q(1) = 0$, $q(2) = 0$, $q(7) = 1$, and $q(3)$ to $q(6)$ are chosen to make I indifferent concerning what to call if he has category 2:

$$\begin{aligned} (1 - q(3))(120/216) &= (1 - q(4))(180/216) \\ &= (1 - q(5))(200/216) \\ &= (1 - q(6))(215/216) = 1 - .6473 \end{aligned}$$

which gives $q(3) = .3651$, $q(4) = .5768$, $q(5) = .6191$, and $q(6) = .6457$. On the second round, II should try to improve the roll except that with two pair he should leave one pair and reroll three dice. Then, II should tell the truth if possible unless he has accepted a truthful claim of 2 or 1. If he has accepted a truthful claim of 2, he should use the Q of (4.1). If he has accepted a truthful claim of 1, he may use I's strategy of the first round, since this never requires a claim of 1.

Sketch of proof: One may check that if I uses the indicated strategy, then II is indifferent as whether to challenge a 3 4 5 or 6. Thus assuming that II challenges 3 through 7, we find that the probability that I wins is $P(\text{I rolls } 1)P(\text{II rolls 1 or 2}) + P(\text{I rolls 2 and calls 2})(1\text{-val}(G(2,2))) + P(\text{I rolls 3 through 7}) = (720/7776)(4320/7776) + (3600/7666)(.388)(1 - .6473) + 3456/7776 = .55919\ldots$. If II uses his indicated strategy, one may also check that I's indicated strategy is optimal against this. ◇

5. Continuous Improvable Liar's Dice: Player I starts as the roller. He observes $X(1)$ from a uniform distribution on the interval $(0,1)$ and claims $y(1) \in (0,1)$. Player II hearing $y(1)$ but not knowing $X(1)$ accepts or challenges. If II challenges, he wins if $X(1) < y(1)$ and loses otherwise. If he accepts, the game is continued with the roles of the players reversed. This time, however, a number $X(2)$ is chosen from the uniform distribution on $(X(1),1)$, and the roller must claim a number $y(2)$ in the interval $(y(1),1)$. This continues *ad infinitum* the game ending after the first challenge, the roles of the players reversing after every acceptance, and $X(n+1) \in \mathcal{U}(X(n),1)$ and $y(n+1) \in (y(n),1)$ for all n. If the game continues forever with neither player challenging, the game is declared a draw although, as we shall see, the probability that this occurs under optimal play of either player is zero. A change of location or scale does not change the problem, so we may as well assume that each $X(n) \in \mathcal{U}(0,1)$ and that the roller must choose $y(n+1)$ from the interval $(z(n),1)$, where $z(n) = (y(n) - X(n))/(1 - X(n))$. Let us denote by $G(z)$ the basic game in which the roller observes $X \in \mathcal{U}(0,1)$ and must choose $y \in (z,1)$. Let $\varphi(z)$ denote the probability that I wins $G(z)$ under optimal play by both players. The original game is $G(0)$, so the value of the original game is $\varphi(0)$. We first present a few lemmas on $\varphi(z)$.

Lemma 1: $\varphi(z)$ is nonincreasing in z, and $\varphi(z) \le 1 - z$.

Proof: Suppose $z' < z$. Then, any strategy available to I in the game $G(z)$ is also available to I in the game $G(z')$, and it guarantees him the same amount. Hence, $\varphi(z') > \varphi(z)$. Moreover, II can guarantee $1 - z$ by challenging any claim I may make. Then I wins if and only if he tells the truth, which occurs with probability $P(X > z) = 1 - z$. ◇

Lemma 2: Let V and W satisfy

$$V = (1-V)^{(1+V)/2} = .544658\ldots \tag{5.1}$$
$$W = 1 - V^2/(1-V) = .348507\ldots$$

Then $\varphi(z) \ge V$ for $0 \le z \le W$. Player I can guarantee this amount by using the strategy: if $X = x$, claim $y = f(x)$ where

$$\begin{aligned} f(x) &= [x + V + V^2(1-x)^{-1/V}]/(1+V) \quad \text{for } 0 < x < W \\ &= x \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{for } W < x < 1. \end{aligned} \tag{5.2}$$

Proof: Suppose $z = 0$. Clearly, II should accept a claim of $y < V$. Suppose II hears a claim of $Y = y > V$ and that $y = f(x)$ where $0 < x < W$. Then the probability that I is telling the truth is $g(y)$, where

$$g(y) = P(X = y | Y = y) = f'(x)/(f'(x) + 1).$$

If II challenges y, then $P(\text{I wins}|Y = y) = g(y)$. If II accepts y, then $P(\text{I wins}|Y = y) = g(y)(1-\varphi(0)) + (1-g(y))(1-\varphi((y-x)/(1-x)))$. Hence it is optimal for II to challenge y if and only if

$$g(y) \le g(y)(1-\varphi(0)) + (1-g(y))(1-\varphi((y-x)/(1-x)))$$

or, equivalently,

$$\varphi(0)f'(x) \le 1 - \varphi((y-x)/(1-x)).$$

Suppose $\varphi(0) < V$. Then, using $f'(x) = (f(x)-x)/((1-x)V)$ for $0 \le x \le W$,

$$\begin{aligned}\varphi(0)f'(x) &= \varphi(0)(y-x)/((1-x)V)\\ &< (y-x)/(1-x)\\ &\le 1-\varphi((y-x)/(1-x))\end{aligned}$$

from Lemma 1. This implies it is optimal for II to accept a $y < V$ and to challenge a $y > V$, which gives

$$\begin{aligned}P(\text{I wins}) &= (V-W)(1-\varphi(0)) + (1-V)\\ &= 1 - W - (V-W)\varphi(0).\\ &= [V^2 - (2V-1)\varphi(0)]/(1-V) \qquad (5.3)\\ &> \varphi(0)[V-(2V-1)]/(1-V) = \varphi(0).\end{aligned}$$

But this indicates that I can guarantee himself more than the value of the game. This contradiction shows that $\varphi(0) \ge V$. Since I's strategy can be used for $z \le W$ to guarantee the same amount, we also have $\varphi(z) \ge V$ for $0 \le z \le W$. ◇

Lemma 3: $\varphi(z) \ge V$ for $0 \le z \le 1-V$. Player I can guarantee this amount by using the strategy: if $X = x$, claim $y = f(x)$, where

$$\begin{aligned}f(x) &= [x + V + V^2(1-x)^{-1/V}]/(1+V) && \text{for } 0 \le x < W, \qquad (5.4)\\ &= (1-V)(x-W)/V + 1 - V && \text{for } W < x < 1-V\\ &= x && \text{for } 1-V < x < 1.\end{aligned}$$

Proof: Suppose $z = 0$ and II hears a claim of $Y = y$ in $(1-V, V)$. Then either $Y = y$ or $X = x$ where x is in $(W, 1-V)$ and $f(x) = y$, the probability of the former being

$$P(X = y|Y = y) = f'(x)/(f'(x)+1) = 1 - V.$$

Thus, if II accepts I's claim, then

$$\begin{aligned} P(\text{I wins}|Y=y) &= (1-V)(1-\varphi(0)) + V(1-\varphi((y-x)/(1-x))) \\ &\le (1-V)(1-V) + V(1-V) = 1-V \end{aligned}$$

since $(y-x)/(1-x) = (2V-1)/V = .16398\ldots$, independent of y, and from Lemma 2, $\varphi(0) > V$ and $\varphi(.16398...) > V$. If II challenges, then $P(\text{I wins}|Y = y) = 1-V$ for all y, so II may as well accept all y in $(1-V, V)$. Now follow the rest of the proof of Lemma 2, the only change being that the first line of (5.3) becomes

$$P(\text{I wins}) \ge (V-W)(1-\varphi(0)) + (1-V).$$

Since I's strategy can be used for all $z < 1-V$, we have $\varphi(z) \ge V$ for all z in $(0, 1-V)$. ◇

Theorem 4: The game $G(z)$ has value $\varphi(z) = min\{V, 1-z\}$ where V satisfies (5.1).

Case 1: If $z < 1-V$, each player has an optimal strategy independent of z. The roller has an optimal pure strategy: If $X = x$, claim $y = f(x)$ where f is given by (5.4). (I may better take advantage of poor play by II in $G(z)$ by telling the truth whenever $X > max\{z, W\}$.) An optimal strategy for the challenger is to accept a claim of y with probability $p(y)$, where

$$\begin{aligned} p(y) &= 1 \text{ for } 0 \le y \le V, \text{ and} \\ &= (1-f^{-1}(y))^{1/V} \text{ for } V \le y \le 1. \end{aligned} \tag{5.5}$$

Case 2: If $z \ge 1-V$, it is optimal for the challenger to challenge always, and for the roller to claim $y = f(X)$ provided $y > z$, and to claim any $y > z$ otherwise.

Proof: First, suppose $z \ge 1-V$ (Case 2). Player II can guarantee losing at most $1-z$ by challenging always (Lemma 1). On the other hand, if player I claims $y = f(X)$ provided $f(X) > z$, then II would be indifferent between challenging and accepting if it were known that $f(X) > z$; but since there is an additional chance that I is lying, II may as well always challenge. Thus, I is guaranteed at least $1-z$, showing $\varphi(z) = 1-z$.

Next, suppose $z < 1-V$ (Case 1). We have seen (Lemma 3) that the indicated strategy for I guarantees him at least V. Assume then that player II uses the strategy $p(y)$ of (5.5). We complete the proof by showing that II's expected loss is at most V. We find I's best response to $p(y)$.

Suppose I observes $X = x \le W$. If I lies, he should announce that y in $(V, 1)$ to achieve the maximum of $U(y) = p(y)(1-\varphi((y-x)/(1-x)))$. For $(y-x)/(1-x) > 1-V$, this becomes $U(y) = p(y)(y-x)/(1-x)$. Then since $p'(y) = -p(y)/(y - f^{-1}(y))$, we find that $U'(y) = p(y)(x - f^{-1}(y))/((y - f^{-1}(y))(1-x))$. Thus among $y \geq x + (1-x)(1-V)$, I's return is maximized by $y = f(x)$ and the value of the return there is $U(f(x)) \geq U(f(1)) = 1 - V$. Among $y < x + (1-x)(1-V)$, the return is less than $p(y)(1-V) \leq 1 - V$, so I's optimal response is $y = f(x)$. (Truth also gives $1 - \varphi(0) \leq 1 - V$.)

Suppose I observes $W < x \leq V$. Now lying at $y \geq x + (1-x)(1-V)$ is maximized at $y = 1$, with return equal to $1 - V$. Truth also gives at most $1 - V$ as does lying at $y < x + (1-x)(1-V)$. So I's optimal response is $y = 1$.

Suppose I observes $V < x \leq 1$. Truth returns $(1 - p(x)) + p(x)(1 - \varphi(0)) = 1 - p(x)\varphi(0)$. Lying should be done at $y = 1$ with return $1 - V$. Thus the optimal return is $max\{1 - V, 1 - p(x)\varphi(0)\}$.

Combining these, the total optimal expected return to I is

$$\int_0^W p(f(x))(f(x)-x)/(1-x)dx + (V-W)(1-V) + \int_V^1 max\{1-V, 1-p(x)\varphi(0)\}dx.$$

Since $(f(x) - x)/(1 - x) = Vf'(x)$, the first term is equal to $V \int_V^1 p(y)dy$. Since $\varphi(0) \geq V$, the third term is less than or equal to $\int_V^1 (1 - p(x)V)dx$. Hence, I's optimal expected return is bounded above by

$$V \int_V^1 p(x)dx + (V-W)(1-V) + \int_V^1 (1-p(x)V)dx = (V-W)(1-V) + (1-V) = V.$$

Since from Lemma 3, I can achieve this amount, II's strategy is optimal. ◇

6. Liar's Coins: In some liar's dice games, the player who accepts a call and rerolls some of the dice is allowed to hide from his opponent which dice are rerolled. This allows the possibility for a player to deliberately try to achieve a low roll by rerolling only those dice that make the roll good in hopes that the opponent will accept the claim and have a hard time improving. We investigate the question of whether this represents a reasonable strategy. To simplify the discussion, we replace poker-type ordering of rolls by the ranking due to the total sum of the dice, and then to simplify further, we replace the dice by coins. The resulting game is as follows.

There are n fair coins and Player I starts by tossing all coins and observing $X(1)$, the total number of heads, and claiming that he sees $y(1)$ heads where $y(1) \geq X(1)$. Player II then challenges or accepts. If II challenges, the game ends in the usual manner. If II accepts, the roles of the players are reversed with II retossing any subset of the coins he wishes. However, II does not show I which coins are retossed. Then, II observes $X(2)$ heads and claims that he sees $y(2)$ heads where $y(2) \geq X(2)$ and $y(2) > y(1)$. Play continues in like manner until one of the players challenges.

First we argue that when n is large it pays occasionally to "go-for-low" by rerolling all of the heads. Except for the first round when all coins must be tossed, the player with control of the coins should either toss all the heads to *destroy* the hand, or all the tails to *improve* it; tossing any other subset of the coins is never of any use. Let x denote the proportion of coins showing heads. By the central limit theorem, if all tails are tossed, the proportion of heads after the toss will be $.5 + x/2 \pm \epsilon$, where ϵ is a term of order $1/\sqrt{n}$. If all heads are tossed, then after the toss there will be a proportion of $x/2 \pm \epsilon$ heads. To simplify things, we take n so large that the ϵ term can be neglected entirely. Assume the controlling player tosses all heads to destroy the hand, and claims $.5 + x/2$. If this claim is believed, the believing player can improve the hand to at most $.5 + x/4$ and will be forced to lie; the other player will win automatically by challenging the claim.

Now, we compute the value, v, of this game, the probability that the controlling player wins under optimal play. The controlling player can either improve the hand or destroy it, in either case claiming $.5 + x/2$. The other player can either believe the claim or challenge it. This leads to a 2×2 matrix game. If the controlling player destroys the hand, he will win if the claim is accepted and lose if it is challenged. If the controlling player improves the hand and is challenged, again he wins. If he improves the hand and is believed, his opponent will win with probability v and so he will win with probability $1 - v$. This results in the following matrix.

	accept	challenge
improve	1 - v	1
destroy	1	0

The value of this game is $1/(1 + v)$ which, equated to v, gives $v = (\sqrt{5} - 1)/2 = .618\ldots$, the golden ratio, as the value. The optimal strategies are that the controlling player should improve with probability v and destroy with probability $1 - v$, and his opponent should accept with probability v and challenge with probability $1 - v$. Since the first player must toss all the coins, the second player can believe all but outrageous claims in the first round and become the controlling player for the next round. Thus, under optimal play the first player wins with probability $1-v = .382\ldots$ This is the only game we treat that favors the second player.

Just how many coins are needed to exhibit the phenomenon of going-for-low? The answer is three! Consider the liar's coins game with $n = 3$. Let $P(a, b)$ denote the probability that the player controlling the coins (about to toss any subset) will win under optimal play, given he has a heads and must claim more than b heads. Then, it is straightforward to compute

$$P(2,2) = 1/2$$
$$P(1,2) = 1/4$$
$$P(0,2) = 1/8$$
$$P(0,1) = 1/2.$$

We now compute $P(1,1)$. The controlling player will claim three heads if and only if he has it, since his opponent is certain to challenge any claim of three heads. Thus, the controlling player has two strategies, improve or destroy, in each case claiming three heads if he has it and two heads otherwise; his opponent has two strategies also, accept a claim of two heads or challenge it. We are led to the matrix game below.

$$\begin{array}{lcc} & \text{accept a two} & \text{challenge} \\ \text{improve} & 1/4+(1-P(2,2))/2+(1-P(1,2))/4 & 3/4 \\ \text{destroy} & (1/2)(1-P(1,2))+(1/2)(1-P(0,2)) & 0 \end{array}$$

$$= \begin{bmatrix} 11/16 & 3/4 \\ 13/16 & 0 \end{bmatrix}$$

The value of this game is $P(1,1) = 39/56$. The controlling player should improve with probability $13/14$ and destroy with probability $1/14$. His opponent should accept a two with probability $6/7$ and challenge it with probability $1/7$.

Finally we can determine the best strategy for the first round and compute the value of the game. Clearly if the first roll is a two or a three, the first player's claim is the same as his roll. He has a choice of lying schemes for outcomes of 0 and 1 however. He can either claim 1 when he has 0 and sometimes claim 2 when he has 1, or claim 2 when he has 0 and sometimes claim 2 when he has 1. From previous solutions it seems that the first strategy will be better, so we solve using it and hope to show later that any change from it is damaging to his chance of winning. Assuming this strategy, $2/3$ of the time with a 1 he should claim a 2, and the other $1/3$ of the time he should claim a 1. With a 0 he should always claim a 1. The opponent should reject all claims of 3, accept a claim of 2 with probability $17/42$, and accept all claims of 0 or 1. The final step is to demonstrate the validity of our previous assumption on the first player's strategy. This is done by showing that claiming 2 with 0 heads gains nothing over claiming 1 head with 0. The value of this game is $807/1344$.

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