LOSE A DOLLAR OR DOUBLE YOUR FORTUNE

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1. Summary and introduction

A gambler with initial fortune x (a positive integer number of dollars) plays a sequence of identical games in which he loses one dollar with probability π, $0 < \pi < 1$, and doubles his fortune with probability $1 - \pi$. Playing continues until, if ever, the gambler is ruined (his fortune drops to zero). Let q_x denote the probability that the gambler starting with initial fortune x will eventually be ruined. Then, q_x satisfies the difference equation

$$q_x = \pi q_{x-1} + (1 - \pi) q_{2x}, \qquad x = 1, 2, \cdots, \tag{1.1}$$

with boundary conditions $q_0 = 1$ and $\lim_{x \to \infty} q_x = 0$. In Section 2, a solution to this difference equation subject to these boundary conditions is explicitly exhibited, and it is shown that there is only one such solution. In Section 3, the equation is extended to allow arbitrary noninteger values for the fortune, and again a solution is found. Section 4 contains several other extensions.

Equation (1.1) arises in connection with the following more general gambling problem described in [2]. A gambler is confronted with a sequence of games affording him even money bets at varying probabilities of success, $p_1, p_2, \cdots$, chosen independently from a distribution function F known to the gambler. The probability of winning the jth game p_j is told to the gambler after he plays game $j - 1$ and before he plays game j. The gambler must decide how much to bet in the jth game as a function of the past history, his present fortune, and the win probability p_j. He may bet any amount not exceeding his present fortune; however, he must bet at least one dollar on each game (called Model 2 in [2]). The problem of the gambler is to choose a betting system (a sequence of functions $b_1, b_2, \cdots$, where b_j is the amount bet in game j) that minimizes the probability of eventual ruin. Theorems relating to the dynamic programming solution of this problem may be found in Truelove [4].

Let q_x denote the infimum, over all betting systems, of the probability of ruin given the initial fortune x. It was shown in [2] that q_x tends to zero exponentially as x tends to infinity, and it was conjectured that, for some $0 < r < 1$ and $c > 0$,

$$q_x r^x \to c \quad \text{as} \quad x \to \infty. \tag{1.2}$$

The preparation of this paper was supported in part by NSF Grant No. GP-8049.

Under this conjecture, the asymptotic (for large x) form of the optimal betting system was found to be

$$b(p) = \max\left\{1, \frac{\log(1-p) - \log p}{\log r^2}\right\}, \tag{1.3}$$

where r is the unique root between zero and one of the equation

$$\int_0^1 \left[p r^{b(p)} + (1-p) r^{-b(p)}\right] dF(p) = 1. \tag{1.4}$$

In [1], Breiman proved conjecture (1.2) to be valid under the condition that F give no mass to some neighborhood of 1, that is, that there exists an $\varepsilon > 0$ such that $F(1-\varepsilon) = 1$.

There remains the question of the validity of the conjecture (1.2), or more generally the validity of the asymptotic form of the optimal betting function (1.3), when F gives mass to all neighborhoods of 1.

In this paper, the simplest such F is considered, namely, the F that gives mass π to 0 and mass $1 - \pi$ to 1, where $0 < \pi < 1$. Under this F, the form of the optimal betting function for all x is obvious: bet the minimum value 1 when the probability of win p is 0 and bet the maximum value x when p is 1. If p is 0, you lose a dollar; if p is 1, you double your fortune.

This leads to the recurrence equation (1.1) for q_x, with the boundary condition $q_0 = 1$. As noted before, it is known that q_x tends to zero exponentially. We add as a boundary condition the weaker assertion $\lim_{x\to 0} q_x = 0$, that together with $q_0 = 1$ is sufficient to insure the unicity of the solution to (1.1), for integer values of x. However, it is of interest, in connection with the gambling system problem from which equation (1.1) originally arose, to investigate (1.1) for noninteger values of x. This investigation leads to a useful solution for rational x. If the initial boundary conditions on q_x for $0 \leqq x < 1$ are continuous in x, then approximations may be found for q_x for x irrational. Furthermore, it is observed that the conjecture (1.2) is not valid for the boundary condition $q_x = 1$ for $0 \leqq x < 1$.

2. A solution of difference equation (1.1)

We present in Theorem 1, the unique solution to the difference equation (1.1) subject to the boundary conditions $q_0 = 1$ and $\lim_{x\to\infty} q_x = 0$, as a series in $\pi^{2^j x}$ $j = 0, 1, 2, \cdots$, with coefficients depending on π.

The systems of equations (1.1), even with q_0 fixed equal to one, is vastly underdetermined. In fact, the values of q_x for x odd, $x = 1, 3, 5, \cdots$, may be chosen arbitrarily and the system (1.1) will serve only to determine the values of q_x for x even. However, the addition of the single boundary condition $\lim_{x\to\infty} q_x = 0$ determines all the q_x uniquely as follows.

THEOREM 1. *Let $0 < \pi < 1$. There exists a unique solution to the difference equation $q_x = \pi q_{x-1} + (1 - \pi) q_{2x}$, $x = 1, 2, 3, \cdots$, subject to the boundary conditions*

$$q_0 = 1 \tag{2.1}$$

and

$$\lim_{x \to \infty} q_x = 0. \tag{2.2}$$

That solution is

$$q_x = \alpha \sum_{j=0}^{\infty} c_j \pi^{2^j x}, \qquad x = 0, 1, 2, 3, \cdots, \tag{2.3}$$

where the c_j are defined inductively by

$$c_0 = 1, \qquad c_j = \frac{1 - \pi}{1 - \pi^{1 - 2^j}} c_{j-1}, \tag{2.4}$$

and where

$$\alpha = \left(\sum_{j=0}^{\infty} c_j \right)^{-1}. \tag{2.5}$$

PROOF. The following proof of unicity is taken from Theorem 1 of MacQueen and Redheffer [3].

If q_x and q'_x both satisfy (1.1), (2.1), and (2.2), then the difference $d_x = q_x - q'_x$ satisfies (1.1), $d_0 = 0$, and (2.2). Suppose that $q_x \neq q'_x$ for some x, and assume, without loss of generality, that $q_x > q'_x$ for some x. Let $x_0 \geqq 1$ be the largest value of x at which d_x achieves its maximum value, assumed to be positive. Then $q_{x_0 - 1} \leqq q_{x_0}$, and $q_{2x_0} < q_{x_0}$. This contradicts (1.1), thus proving unicity.

To see that (2.3) is a solution, note first from (2.4) that $|c_j| \leqq |c_{j-1}|$ so that (2.3) converges absolutely for $0 < \pi < 1$. Thus, (2.2) is obviously satisfied and (2.1) is satisfied from (2.5). Using (2.3), we compute the right side of (1.1):

$$\begin{aligned} \pi q_{x-1} &+ (1 - \pi) q_{2x} \\ &= \pi \alpha \sum_{j=0}^{\infty} c_j \pi^{2^j (x-1)} + (1 - \pi) \alpha \sum_{j=1}^{\infty} c_j \pi^{2^j 2x} \\ &= \alpha \sum_{j=0}^{\infty} c_j (\pi^{1 - 2^j}) \pi^{2^j x} + \alpha \sum_{j=1}^{\infty} c_{j-1} (1 - \pi) \pi^{2^j x} \\ &= \alpha c_0 \pi^x + \alpha \sum_{j=1}^{\infty} \left(c_j \pi^{1 - 2^j} + c_{j-1} (1 - \pi) \right) \pi^{2^j x}. \end{aligned} \tag{2.6}$$

From the recurrence relation (2.4), this is equal to q_x, thus showing that (1.1) is satisfied and completing the proof.

The solution (2.3) is quite suitable for computational purposes, provided π is not too close to one. Table I, for which I am greatly indebted to David Cantor, gives an indication of the behavior of q_x for moderate values of π. As π tends to one, $q_{z/(1-\pi)}$ tends to $H(z) = \alpha' \Sigma_0^\infty c_j' \exp\{-2^j z\}$, where $c_0' = 1$, $c_j' = \Pi_1^j(1 - 2^i)^{-1}$ and $\alpha' = (\Sigma_0^\infty c_j')^{-1}$. In Section 4.5, it is seen that $1 - H(z)$ is the cumulative distribution function of $Z = \Sigma_0^\infty 2^{-j} Y_j$, where $Y_0, Y_1, Y_2, \cdots$ are independent identically distributed exponential variables.

A simple asymptotic approximation to q_x as π tends to zero is $q_x \sim \pi^x(1 + \pi)$, $x = 1, 2, \cdots$.

TABLE I

TABLE OF q_x

	.5	.6	.7	.8	.9
1	.7044	.8436	.9421	.9895	.9997
2	.4087	.6089	.8071	.9477	.9974
3	.2188	.4043	.6446	.8740	.9901
4	.1131	.2569	.4921	.7802	.9760
5	.0574	.1593	.3649	.6788	.9541
10	.0018	.0129	.0681	.2734	.7577
15	.0001	.0010	.0116	.0952	.5248
20		.0001	.0020	.0318	.3380
25			.0003	.0105	.2096
30				.0035	.1272
35				.0012	.0763
40				.0004	.0455
45				.0001	.0270
50					.0160

3. Fractional fortunes

The problem of the preceding section may be extended to the case in which the initial fortune x may be any nonnegative number. The difference equation to be considered is thus

$$q_x = \pi q_{x-1} + (1 - \pi)q_{2x}, \qquad x \geqq 1. \tag{3.1}$$

If the gambler is declared ruined when his fortune falls below one, then the boundary condition (2.1) is replaced by $q_x = 1$ if $0 \leqq x < 1$. We consider in this section an arbitrary bounded q_x for $0 \leqq x < 1$.

We first note that if q_x satisfies (3.1) and (2.2), then q_x is bounded for $x \geqq 0$. Since $\lim_{x\to\infty} q_x = 0$, there exists an integer N such that $|q_x| < 1$ if $x \geqq N$. From (3.1),

$$|q_{x-1}| = \pi^{-1}|q_x - (1 - \pi)q_{2x}| \leqq \pi^{-1}|q_x| + \pi^{-1}(1 - \pi)|q_{2x}|, \tag{3.2}$$

so that $|q_x| < \pi^{-1}(2 - \pi)$ for $x \geqq N - 1$. By induction therefore, $|q_x| < (\pi^{-1}(2 - \pi))^N$ for $x \geqq 0$.

We next note that probabilistic considerations alone allow us to infer the existence of a solution to (3.1) and (2.2) subject to an arbitrary initial condition on q_x for $0 \leqq x < 1$, provided $0 \leqq q(x) \leqq 1$. Then, since the product of a solution to (3.1) and (2.2) by a scalar, and the sum of two solutions to (3.1) and (2.2) are also solutions to (3.1) and (2.2), we may achieve any bounded initial condition on q_x for $0 \leqq x < 1$.

Finally, we note that an extension of the argument of the previous section allows us to conclude that there exists a unique solution to (3.1) and (2.2) subject to an arbitrary bounded initial condition on q_x for $0 \leqq x < 1$. For if $\hat{q}_x$ represents the difference of any two solutions, then $\hat{q}_x$ satisfies (3.1) and (2.2) and the boundary condition $\hat{q}_x = 0$ for $0 \leqq x < 1$. We must show that $\hat{q}_x$ vanishes for all $x \geqq 1$. Since $\hat{q}_x$ must be bounded, let $\beta = \sup_{x \geqq 0} \hat{q}_x$, suppose, without loss of generality, that $\beta > 0$, and let x_0 be the largest number for which there exists a sequence $x_n \to x_0$ and $\hat{q}_{x_n} \to \beta$. Then equation (3.1) shows that $\hat{q}_{2x_n} \to \beta$, contradicting the assertion that x_0 was the largest number with this property, and completing the proof of unicity.

The following theorem presents a class of solutions to (3.1) and (2.2), but it is not known whether an arbitrary bounded initial condition on q_x, $0 \leqq x < 1$, can be achieved as one of them.

THEOREM 2. *Let $\alpha(x)$ be a bounded periodic function of period one, and let, for $0 < \pi < 1$,*

$$q_x = \sum_{j=0}^{\infty} c_j \alpha(2^j x) \pi^{2^j x}, \qquad x \geqq 0, \tag{3.3}$$

where $c_0 = 1$ and $c_j = (1 - \pi)(1 - \pi^{1-2^j})^{-1} c_{j-1}$ for $j = 1, 2, \cdots$. Then q_x satisfies (3.1) and (2.2).

The proof, being similar to the corresponding part of Theorem 1, is omitted. The hypothesis that α be bounded is equivalent to (2.2).

This theorem may be used to find q_x for all x with dyadic fractional part for an arbitrary boundary condition on q_x for $0 \leqq x < 1$. Such a solution depends only on q_x, $0 \leqq x < 1$, through the dyadic rational values of x. To accomplish this, it is sufficient to find $\alpha(\frac{1}{2})$, $\alpha(\frac{1}{4})$, $\alpha(\frac{3}{4})$, $\cdots$. As in Theorem 1, equation (3.3) with $x = 0$ enables us to find $\alpha(0)$. For $x = \frac{1}{2}$, equation (3.3) becomes

$$q_{1/2} = c_0 \alpha(\tfrac{1}{2}) \pi^{1/2} + \sum_{j=1}^{\infty} c_j \alpha(0) \pi^{2^{j-1}} \tag{3.4}$$

from which $\alpha(\frac{1}{2})$ may be found. For $x = \frac{1}{4}$, equation (3.3) becomes

$$q_{1/4} = c_0 \alpha(\tfrac{1}{4}) \pi^{1/4} + c_1 \alpha(\tfrac{1}{2}) \pi^{1/2} + \sum_{j=2}^{\infty} c_j \alpha(0) \pi^{2^{j-2}}, \tag{3.5}$$

from which $\alpha(\frac{1}{4})$ may be found. It is clear that $\alpha(x)$ may be found for all dyadic rational x by this method.

In addition, one can find $\alpha(x)$, and thus $q(x)$ for rational x directly. We illustrate for $x = \frac{1}{3}$. Evaluating (3.3) at $x = \frac{1}{3}$ and $x = \frac{2}{3}$ yields

$$\begin{aligned} q_{1/3} &= \alpha\left(\tfrac{1}{3}\right) \sum_{j \text{ even}} c_j \pi^{2^j/3} + \alpha\left(\tfrac{2}{3}\right) \sum_{j \text{ odd}} c_j \pi^{2^j/3} \\ q_{2/3} &= \alpha\left(\tfrac{1}{3}\right) \sum_{j \text{ odd}} c_j \pi^{2^{j+1}/3} + \alpha\left(\tfrac{2}{3}\right) \sum_{j \text{ even}} c_j \pi^{2^{j+1}/3} \end{aligned} \tag{3.6}$$

from which one can solve for $\alpha\left(\frac{1}{3}\right)$ and $\alpha\left(\frac{2}{3}\right)$ provided the determinant of this system of equations does not vanish. In this case, it is clear that the determinant is not zero, since in each row the summation over j even is larger than the absolute value of the summation over j odd.

The general problem of finding q_x for x irrational seems more difficult. If the boundary condition on q_x for $0 \leqq x < 1$ specifies a bounded continuous function of x, in particular, if it is assumed that $q_x = 1$ for $0 \leqq x < 1$, then q_x for $x \geqq 1$ has discontinuities only at the dyadic rationals, and hence can be approximated as closely as desired by q_x for x dyadic rational.

To see that if the boundary condition specifies a bounded continuous function of x, then q_x is continuous at all x for which x is not dyadic rational, we may argue as follows. Let q_x be a solution to (3.1) and (2.2), suppose that q_x is continuous for $0 \leqq x < 1$, and let $q_x^+ = \limsup_{t \to x} q_t$ and $q_x^- = \liminf_{t \to x} q_t$ for $x > 0$. Then, for $x > 0$,

$$\begin{aligned} q_x^+ &\leqq \pi q_{x-1}^+ + (1 - \pi) q_{2x}^+, \\ q_x^- &\geqq \pi q_{x-1}^- + (1 - \pi) q_{2x}^-. \end{aligned} \tag{3.7}$$

Hence, letting $d_x = q_x^+ - q_x^-$, we see that

$$0 \leqq d_x \leqq \pi d_{x-1} + (1 - \pi) d_{2x}, \qquad x > 0, \tag{3.8}$$

$$d_x = 0, \qquad 0 < x < 1, \tag{3.9}$$

$$\lim_{x \to \infty} d_x = 0. \tag{3.10}$$

If we consider this equation on the set of all positive x not dyadic rational, then the argument used to prove the unicity of (3.1) may be used here again to show $d_x = 0$ for all x not dyadic rational even though there is an inequality in (3.8) (as in MacQueen and Redheffer [3]) instead of equality. Hence, q_x is continuous at all x not dyadic rational.

Finally, we note that the conjecture (1.2) is false in the present case. In fact

$$\begin{aligned} \frac{q_x}{\pi^x} &= \alpha(x) + \sum_{j=1}^{\infty} \alpha(2^j x) c_j \pi^{2^j x - x} \\ &\sim \alpha(x) \end{aligned} \tag{3.11}$$

which is not constant in x, but periodic of period one. However, the trivially optimal betting system is still of the (asymptotic) form (1.3).

4. Other extensions

The method of expanding q_x as a series in $\pi^{2^j x}$ appears to be quite useful. We mention several other related difference equations that can be solved by the use of such an expansion.

4.1. Given $\pi_{-1} > 0$, $\pi_i \geqq 0$ for $i = 0, \cdots, n$, $\pi_{n+1} > 0$, and $\Sigma_{-1}^{n+1} \pi_i = 1$, find q_x to satisfy

$$q_x = \sum_{i=-1}^{n} \pi_i q_{x+i} + \pi_{n+1} q_{2x}, \qquad x = 1, 2, \cdots, \tag{4.1}$$

subject to

$$q_0 = 1, \qquad \lim_{x \to \infty} q_x = 0. \tag{4.2}$$

The unicity of the solution follows as before. We attempt a series for q_x of the form

$$q_x = \sum_{j=0}^{\infty} c_j r^{2^j x}, \qquad x = 0, 1, 2, \cdots. \tag{4.3}$$

Formal substitution of this series into (4.1) yields

$$\sum_{j=0}^{\infty} c_j r^{2^j x} = \sum_{j=0}^{\infty} c_j \left(\sum_{i=-1}^{n} \pi_i r^{2^j i} \right) r^{2^j x} + \pi_{n+1} \sum_{j=1}^{\infty} c_{j-1} r^{2^j x}. \tag{4.4}$$

We equate the coefficients of $r^{2^j x}$. For $j = 0$, we obtain

$$c_0 = c_0 \sum_{i=-1}^{n} \pi_i r^i. \tag{4.5}$$

This equation in r has a unique root between zero and one, since the right side is convex in r, tends to $+\infty$ as r tends to zero, and to $c_0 \Sigma_{-1}^{n} \pi_i < c_0$ as r tends to one. For arbitrary $j \geqq 1$, we obtain

$$c_j = c_j \sum_{i=-1}^{n} \pi_i r^{2^j i} + c_{j-1} \pi_{n+1}. \tag{4.6}$$

Thus, the solution of (4.1) subject to (4.2) is (4.3), where r satisfies (4.5) and the c_j are defined recursively by (4.6), and where c_0 is chosen to satisfy $q_0 = 1$.

4.2. The model called Model I in [2] leads in the present context to the equation

$$q_x = \pi q_{x-1} + (1 - \pi) q_{2x-1}, \qquad x = 1, 2, \cdots, \tag{4.7}$$

subject to $q_0 = 1$ and $\lim_{x \to \infty} q_x = 0$. Since no new ideas are involved, we omit the details.

4.3. Analogous methods can be used to treat difference equations involving more terms of the form q_{x-n} and correspondingly more boundary conditions.

As an example, consider the equation ($\pi_1 > 0, \pi_2 > 0, \pi_1 + \pi_2 < 1$)

$$q_x = \pi_1 q_{x-1} + \pi_2 q_{x-2} + (1 - \pi_1 - \pi_2) q_{2x}, \qquad x = 1, 2, \cdots, \tag{4.8}$$

subject to specified values of q_0 *and* q_{-1} and subject to $\lim_{x\to\infty} q_x = 0$. The unicity of the solution follows as before. Again we try a solution of the form (4.2). Formal substitution into (4.8) yields

$$\sum_{j=0}^{\infty} c_j r^{2^j x} = \sum_{j=0}^{\infty} c_j (\pi_1 r^{-2^j} + \pi_2 r^{-2^{j+1}}) r^{2^j x} + (1 - \pi_1 - \pi_2) \sum_{j=1}^{\infty} c_{j-1} r^{2^j x}. \tag{4.9}$$

Again, we equate coefficients of $r^{2^j x}$. For $j = 0$, we obtain $1 = \pi_1 r^{-1} + \pi_2 r^{-2}$. There are two roots to this equation,

$$r_1 = \tfrac{1}{2}[\pi_1 + (\pi_1^2 + 4\pi_2)^{1/2}], \qquad r_2 = \tfrac{1}{2}[\pi_1 - (\pi_1^2 + 4\pi_2)^{1/2}]. \tag{4.10}$$

For $j > 0$, we obtain for each r_i a recursion formula for the c_j, denoted by $c_{j,i}$:

$$c_{j,i} = c_{j,i}(\pi_1 r_i^{-2^j} + \pi_2 r_i^{-2^{j+1}}) + c_{j-1,i}(1 - \pi_1 - \pi_2) \tag{4.11}$$

for $j = 1, 2, \cdots$ and $i = 1, 2$. Hence, the general solution of (4.8) subject to $\lim_{x\to\infty} q_x = 0$ is

$$q_x = \sum_{j=0}^{\infty} c_{j,1} r_1^{2^j x} + \sum_{j=0}^{\infty} c_{j,2} r_2^{2^j x}. \tag{4.12}$$

The values of $c_{0,1}$ and $c_{0,2}$ may be chosen to obtain specified values of q_0 and q_{-1}.

4.4. Finally, we note that in expanding q_x in terms of $r^{2^j x}$, the "2" appears because we are concerned with the doubling of the fortune. To solve an equation of the type $q_x = \pi q_{x-1} + (1 - \pi) q_{3x}$ an expansion as a series in $\pi^{3^j x}$ works. The details involve no new ideas. However, it might be worthwhile to give an explicit solution to the equation

$$q_x = \pi_1 q_{x-1} + \pi_2 q_{2x} + \pi_3 q_{3x}, \qquad x = 1, 2, \cdots, \tag{4.13}$$

where $\pi_1 > 0$, $\pi_2 > 0$, $\pi_3 > 0$, and $\pi_1 + \pi_2 + \pi_3 = 1$ subject to the condition $\lim_{x\to\infty} q_x = 0$. This equation requires an expansion of the form

$$q_x = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{i,j} \pi_1^{2^i 3^j x}, \qquad x = 0, 1, 2, \cdots. \tag{4.14}$$

By substituting this expansion into (4.13) and equating coefficients of $\pi_1^{2^i 3^j}$, we find the following recurrence relations for the $c_{i,j}$,

(4.15)
$$c_{i,j} = \frac{c_{i-1,j}\pi_2 + c_{i,j-1}\pi_3}{1 - \pi_1^{1-2^i 3^j}}, \qquad i = 1, 2, \cdots, \quad j = 1, 2, \cdots,$$

$$c_{i,0} = \frac{c_{i-1,0}\pi_2}{1 - \pi_1^{1-2^i}}, \qquad i = 1, 2, \cdots,$$

$$c_{0,j} = \frac{c_{0,j-1}\pi_3}{1 - \pi_1^{1-3^j}}, \qquad j = 1, 2, \cdots.$$

The value of $c_{0,0}$ may be chosen to obtain a specified value of q_0.

4.5. Let $Y_0, Y_1, Y_2, \cdots$ be independent identically distributed random variables with exponential densities

$$f(y) = \begin{cases} e^{-y}, & y \geqq 0, \\ 0, & y < 0. \end{cases} \tag{4.16}$$

Let $0 < \beta < 1$, and let $Z = \Sigma_0^\infty \beta^j Y_j$. We find the distribution function of Z, $G(z) = P(Z \leqq z)$, as follows. Note that Z may be written as $Z = Y_0 + \beta Z_1$, where $Z_1 = Y_1 + \beta Y_2 + \cdots$ has the same distribution as Z, and is independent of Y_0. Therefore, for $z > 0$,

$$\begin{aligned} G(z) &= \int_0^z e^{-y} G\left(\frac{z-y}{\beta}\right) dy \\ &= \beta \int_0^{z/\beta} e^{-z+\beta x} G(x)\, dx. \end{aligned} \tag{4.17}$$

Multiplying both sides by e^z and differentiating with respect to z, yields

$$G(z) + G'(z) = G\left(\frac{z}{\beta}\right). \tag{4.18}$$

We solve this equation subject to the boundary conditions

$$G(0) = 0, \qquad \lim_{z\to\infty} G(z) = 1, \tag{4.19}$$

and subject to the existence of one continuous derivative of G. (In the present case G, being the convolution of an infinite number of absolutely continuous random variables, must be C^∞.) The unicity of the solution to (4.18) subject to (4.19) and the continuity of G' follows as in the previous problems. It is then a straightforward matter to check that the expansion

$$G(z) = 1 - \alpha \sum_{j=0}^{\infty} c_j \exp\{-\beta^{-j} z\}, \tag{4.20}$$

where

$$c_0 = 1, \qquad c_j = \prod_{i=1}^{j} (1 - \beta^{-i})^{-1}, \qquad \alpha = \left(\sum_{j=0}^{\infty} c_j\right)^{-1}, \tag{4.21}$$

satisfies (4.18) and (4.19), and thus represents the distribution function of Z.

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The author gratefully acknowledges useful discussions on this problem with David Cantor and James B. MacQueen.

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