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Part I

Stopping games

Chapter 1

Selection by Committee

Thomas S. Ferguson

ABSTRACT The many-player game of selling an asset, introduced by Sakaguchi and extended to monotone voting procedures by Yasuda, Nakagami and Kurano, is reviewed. Conditions for a unique equilibrium among stationary threshold strategies are given for the independent, unanimous consent case.

1.1 Introduction

Decision making by committee is an important aspect of the Bayesian viewpoint of decision theory. Such problems are essentially game-theoretic in nature. Here we consider the selection by a committee of a candidate for a position. This is a game-theoretic version with many players of the well-known problem of selling an asset, also called the house-hunting problem, introduced by MacQueen and Miller [6], Derman and Sacks [3] and Chow and Robbins [1].

A committee is charged with the duty of selecting a candidate for a position. Each member of the committee has his/her own way of viewing a candidate's worth, which may be related or somewhat opposed to the viewpoints of the other members. Candidates appear sequentially and are voted on by the committee. A candidate once rejected cannot be recalled. This is modelled as a multistage game with many players.

The study of this problem was initiated by Sakaguchi [7] for the 2-person unanimous consent case, and extended to allow Poisson arrivals and more than one selection in Sakaguchi [8, 9]. These ideas were extended to many players with a majority rule in Kurano et al. [5], and then to arbitrary voting rules (simple games) in Yasuda et al. [11]. Szajowski and Yasuda [10] consider the problem when the payoffs are functions of a homogenious Markov chain.

These papers are concerned mainly with the existence of an equilibrium. The question of multiple equilibria is not considered. The impression is given that the equlibrium is unique. Surprisingly, this is not the case.

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[0] *AMS Subject Classifications.* 60G40, 90 G 39, 90D45.

Key words. selling an asset game, monotone voting procedures, threshold strategies.

Simple examples are given below showing that more than one subgame perfect equilibrium may occur. After setting up the problem in Section 2, two theorems are given. The first relates equilibrium payoffs with equilibrium strategies. The second gives necessary and sufficient conditions for an equilibrium, based on equilibrium equations of Sakaguchi. In Section 4, a theorem is presented giving general conditions under which there exists a unique equilibrium among stationary threshold strategies for the independent, unanimous consent case. Some surprising results for the exponential distribution are contained in the last section.

1.2 The Problem

Let m denote the number of committee members, referred to below as players, and let M denote the set of all players $M = \{1, \dots, m\}$. The players sequentially observe i.i.d. m-dimensional vectors, $\boldsymbol{X}_1, \boldsymbol{X}_2, \dots$, from a known distribution $F(\boldsymbol{x})$ with finite second moments, $E(|\boldsymbol{X}|^2) < \infty$. After each observation, the players vote on whether or not to stop and accept the present observation or to continue observing. The players have possibly differing costs of observation. Let $\boldsymbol{c} = (c_1, \dots, c_m)$ denote the m-vector of costs, with Player i paying c_i for each observation. If at stage n, the players vote to accept the present observation, $\boldsymbol{X}_n = (X_{1n}, \dots, X_{mn})$, then the payoff to Player i is $X_{in} - nc_i$ for $i = 1, \dots, m$.

The voting decision is made according to the rules of a simple game. A coalition is a subset of the players. Let $\mathcal{C} = \{C : C \subset M\}$ denote the class of all coalitions. A simple game is defined by giving the characteristic function, $\phi(C)$, which maps coalitions $C \in \mathcal{C}$ into the set $\{0, 1\}$. Coalitions C for which $\phi(C) = 1$ are called winning coalitions, and those for which $\phi(C) = 0$ are called losing coalitions. Let $\mathcal{W} = \{C : \phi(C) = 1\}$ denote the class of winning coalitions and $\mathcal{L} = \{C : \phi(C) = 0\}$ denote the class of losing coalitions. These are assumed to satisfy the properties, (1) $M \in \mathcal{W}$, (2) $\emptyset \in \mathcal{L}$ and (3) monotonicity: $T \subset S \in \mathcal{L}$ implies $T \in \mathcal{L}$; namely, subsets of losing coalitions are losing. This implies that supersets of winning coalitions are winning.

A strategy for each player is a voting rule at each stage based on all past information. As in Sakaguchi, we restrict attention to stationary threshold rules. For player i, such a rule is determined by a number a_i, with the understanding that at stage n player i votes to accept $\boldsymbol{X}_n$ if $X_{in} > a_i$.

Given the vector of thresholds, $(a_1, \dots, a_m)$, and an observation, $\boldsymbol{X} = (X_1, \dots, X_m)$, we denote the stopping set by $A = A_{a_1, \dots, a_m}(\boldsymbol{X})$. This set may be decomposed into a union of disjoint sets, one for each winning coalition.

$$A = \bigcup_{C \in \mathcal{W}} \left[\bigcap_{i \in C} \{X_i \geq a_i\} \right] \cap \left[\bigcap_{i \in C^c} \{X_i < a_i\} \right] \tag{1.1}$$

Let $V_i = V_i(a_1, \ldots, a_m)$ denote player i's expected return from such a joint strategy. If $P(A) = P(A(\boldsymbol{X})) = 0$, then play never stops and $V_i(A) = -\infty$ for all i. If $P(A) > 0$, we may compute V_i from the optimality equation,

$$V_i = -c_i + E[X_i I(A)] + V_i(1 - P(A)),$$

where $I(A)$ represents the indicator function of the set A. This may be solved to find

$$V_i = (E[X_i I(A)] - c_i)/P(A) \qquad \text{for } i = 1, \ldots, m. \tag{1.2}$$

We analyze this as a noncooperative game. We assume that $E|\boldsymbol{X}|^2 < \infty$. Under this assumption, for each player there is an optimal reply among *all* stopping rules to any stationary strategy choices of the other players and it may be chosen to be a stationary threshold rule. The weaker assumption that $E|\boldsymbol{X}| < \infty$ may be made but it requires a restriction that the jointly chosen stopping rule be such that the expectation of negative parts of all the returns be greater than $-\infty$. With the stronger assumption, all stopping rules are allowed, even rules that do not stop with probability one.

We seek Nash equilibria, that is, we seek vectors $(a_1, \ldots, a_m)$ such that

$$V_i(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_m) = \sup_a V_i(a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_m) \tag{1.3}$$

for all $i = 1, \ldots, m$. When $m \geq 2$, there may be many equilibria with the property that $P(A) = 0$ where all players receive $-\infty$. (For example when unanimity is required, if any two players refuse to stop, this is an equilibrium no matter what the other players do.) This is the worst of all possible equilibria, and it is not a (trembling hand) perfect equilibrium. Since such equilibria are trivial and not interesting, we restrict attention only to equilibria for which $P(A) > 0$.

1.3 The Equilibrium Equations

With a committee of a single person ($m = 1$), the problem is known as the house-hunting problem or the problem of selling an asset, and has been studied extensively. It is may be considered as a simple example of the general theory of optimal stopping as treated in Chow, Robbins and Siegmund [2] or Ferguson [4]. Several results from the theory of the house-hunting problem carry over to the problem of decision by committee. The proofs of the following theorems use the same arguments as for the house-hunting problem, but require the general theory.

Theorem 1.1. *If* $\boldsymbol{a} = (a_1, \dots, a_m)$ *is an equilibrium vector with finite equilibrium payoff* $\boldsymbol{V} = (V_1, \dots, V_m)$*, then* $\boldsymbol{V}$ *is an equilibrium vector with the same equilibrium payoff.*

We would like to say that in any equilibrium, $\boldsymbol{a} = \boldsymbol{V}$. But there may exist an $\epsilon > 0$ such that $P(V_i - \epsilon < X_i < V_i + \epsilon) = 0$ in which case, any a_i in the interval $(V_i - \epsilon, V_i + \epsilon)$ gives the same equilibrium payoff vector. All we can say is that we may choose a_i equal to V_i.

Proof: Consider Player 1. With $a_2, \dots, a_m$ fixed, this is an optimal stopping problem. The general theory (see Ferguson [4], Chapter 3 Section2, and Chapter 4 Section 1) says that under the condition $EX_1^2 < \infty$, an optimal stopping rule exists (and is of the threshold type) and it is given by the principle of optimality. Let V_1 denote the optimal return for this problem. The principle of optimality says that if you pay c_1 and observe $X_1 < V_1$ you may as well continue, since you can obtain V_1 by continuing. If you observe $X_1 > V_1$, you may as well stop since by continuing, the best you can receive is V_1. Thus, you may as well take $a_1 = V_1$. The same argument works for the other players. ■

When $P(A) > 0$, the equilibrium equations determine the equilibria.

Theorem 1.2. *At any equilibrium with* $P(A) > 0$ *and* $a_i = V_i$ *for* $i = 1, \dots, m$*, we have*

$$E[(X_k - a_k)I(A)] = c_k \qquad \text{for } k = 1, \dots, m. \tag{1.4}$$

Conversely, if $P(A) > 0$ *and (1.4) is satisfied, then* $\boldsymbol{a}$ *is an equilibrium with* $V_i = a_i$ *for all* i*. All such equilibria are sub-game perfect.*

Proof. If $P(A) > 0$, then (1.4) follows by substituting $V_i = a_i$ in (1.2). Conversely, suppose $P(A) > 0$ and (1.4) is satisfied, and consider Player k. As in the proof of 1.1, the general theory says that an optimal threshold a_k^* exists and is equal to the return V_k and so satisfies equation (1.4) when the other a_i for $i \neq k$ are fixed. We must show unicity of the solution. To do this we show that the left side of (1.4) is continuous in a_k with the other a_i, $i \neq k$ fixed, and strictly decreasing from ∞ to 0 and possibly to $-\infty$. In the left side of (1.4), we substitute A of (1.1), and separate the summation over $\mathcal{W}$ into three classes, those sets C that contain k and are losing without k, those that contain k and are winning without k, and those that do not contain k.

$$E[(X_k - a_k)I(A)] = \sum_{\substack{C \in \mathcal{W} \\ k \in C \\ C\setminus\{k\} \notin \mathcal{W}}} E\big[(X_k - a_k)\prod_{i \in C} I(X_i > a_i)\prod_{i \notin C} I(X_i \le a_i)\big]$$
$$+ \sum_{\substack{C \in \mathcal{W} \\ k \notin C}} E\big[(X_k - a_k)\prod_{i \in C} I(X_i > a_i)\prod_{\substack{i \notin C \\ i \neq k}} I(X_i \le a_i)\big]. \tag{1.5}$$

This follows since monotonicity implies that if $C \in \mathcal{W}$ and $k \notin C$, then $C \cup \{k\} \in \mathcal{W}$. The last summation is either zero (if k is in every winning coalition) or linear in a_k and strictly decreasing from $+\infty$ to $-\infty$. The next to last summation is either zero (if k is a dummy, i.e. if the summation is empty) or strictly decreasing from $+\infty$ to 0. Not both summations can be zero. It is also clear that this is continuous in a_k. Thus for each k there exists a unique solution for a_k of equation (1.4) with the other a_i, $i \neq k$, fixed. ■

The existence of equilibria for general simple games has been proved by Yasuda et al. [11]. Here is an example to show that there may be more than one equilibrium even if the X_i are independent. We take $m = 2$ and X_1 and X_2 independent, with X_1 being uniform on the interval (0,1), and X_2 taking the value 0 with probability 5/6 and the value 1 with probability 1/6. For both players the cost of observation is 1/8. The voting game is taken to be unanimity; both players must agree on the candidate. There are two perfect equilibria in threshold strategies. In the first, Player 2 votes to accept every candidate, and Player 1 only votes for candidates such that $X_1 > 1/2$. The equilibrium payoff is $(V_1, V_2) = (1/2, -1/12)$. In the second equilibria, Player 1 votes to accept every candidate, and Player 2 only votes for candidates such that $X_2 = 1$. This has equilibrium payoff $(V_1, V_2) = (-1/4, 1/4)$. Clearly Player 1 prefers the first equilibrium and Player 2 the second.

The last section contains other examples with more than one equilibrium in which the X_i are i.i.d. There are similar examples with many equilibria, but in all of them the equilibrium vectors are noncomparable; that is to say, if Player 1 prefers one of two equilibria, the Player 2 prefers the other.

1.4 Uniqueness of Equilibria in the Independent, Unanimous Consent Case

We assume from here on that the X_i are independent, and that unanimous consent is required to accept a candidate. We give a simple condition for the existence of a unique equilibrium in threshold strategies. Under the unanimous consent voting rule, the set A of (1.1) becomes simply

$$A = \bigcap_{i \in M} \{X_i > a_i\}, \tag{1.6}$$

and in the independent case, the equilibrium conditions (1.4) become

$$E[(X_k - a_k)|X_k > a_k]P(A) = c_k \qquad \text{for } k = 1, \ldots, m. \tag{1.7}$$

Definition 1.1. *A random variable X is said to have strictly decreasing residual expectation if $E[(X - a)|X > a]$ is a strictly decreasing function of a from ∞ when $a = -\infty$, to 0 when a is equal to the right support point of X.*

Without the word "strictly", this is called "New better than used in expectation". Note that when X has decreasing residual expectation, then X is a continuous random variable. This is because if X gives positive mass to a point x_0, then $E[(X - a)|X > a]$ has a jump up at the point x_0. More generally, one can show that if X has decreasing conditional expectation, then X has decreasing density on its support, and its support is an interval (extending possibly to $+\infty$ but not to $-\infty$). In particular, $E((X - a)|X > a)$ is a continuous function of a on the support of X.

Theorem 1.3. *In the unanimous consent case, if the X_i are independent and have strictly decreasing residual expectation, then there exists a unique equilibrium in threshold strategies.*

Proof: From (1.7), all threshold equilibria must satisfy

$$\frac{1}{c_k} E(X_k - a_k | X_k > a_k) = \frac{1}{P(A)} \qquad \text{for } k = 1, \ldots, m. \tag{1.8}$$

In particular, this means that all $E(X_k - a_k|X_k > a_k)/c_k$ are equal. From the hypothesis of strictly decreasing residual expectation, we may find for each θ sufficiently large and each $k = 1, \ldots, m$, a unique number $a_k(\theta)$ such that

$$\frac{1}{c_k} E(X_k - a_k(\theta)|X_k > a_k(\theta)) = \theta.$$

As θ decreases, each $a_k(\theta)$ increases strictly and continuously, until one or perhaps several of the $a_k(\theta)$ reach the upper bound of the support of X_k. But as this occurs, $P(A(\theta))$ decreases strictly and continuously to zero. Therefore there exists a unique value θ_0 such that $\theta_0 = 1/P(A(\theta_0))$. ■

1.5 The Exponential Case

The exponential distribution is on the boundary of the set of distributions with decreasing residual expectation since the residual expectation is constant on the support. Suppose all the X_i have exponential distributions. Since the utilities of the players are determined only up to location and scale, we take without loss of generality all the distributions to have support $(0, \infty)$ and to have mean 1. Then

$$E(X_k - a_k | X_k > a_k) = \begin{matrix} 1 & \text{if } a_k \geq 0, \\ 1 - a_k & \text{if } a_k < 0. \end{matrix}$$

Suppose without loss of generality that the c_i are arranged in nondecreasing order. If there is a unique smallest c_i, that is if $c_1 < c_2$, then there is a unique solution to equations (1.7), and it has the property that a_2 through a_n are negative. If $c_1 < 1$, then a_1 is determined from (1.7) by

$$1 = \frac{c_1}{P(X_1 > a_1)} \qquad \text{or} \qquad a_1 = \log(1/c_1)$$

and for $i = 2, \ldots, n$,

$$1 - a_i = \frac{c_i}{P(X_1 > a_1)} \qquad \text{or} \qquad a_i = -(c_i - c_1)/c_1.$$

In this equilibrium, we see a well-known phenomenon. In committee decisions, the person who values time the least has a strong advantage. This is the committee member who is most willing to sit and discuss at length small details until the other members who have more useful ways of spending their time give in. In this example, Player 1 dominates the committee; all the other members accept the first candidate that is satisfactory to Player 1, who uses an optimal strategy as if the other players were not there. In equilibrium, this committee has the structure of a dictatorship.

If $c_1 > 1$ in this example, then all players accept the first candidate to appear. This is agreeable to all committee members, who are using an optimal strategy as if the other players were not there.

If all c_k are equal and less than 1, any set of $a_i > 0$ such that $\prod_{i=1}^n (1 - F_i(a_i))$ is equal to the common value of the c_k is in equilibrium. We see that without the condition that the distributions have *strictly* decreasing residual expectation, there may be a continuum of equilibria.

The phenomenon of the player with the smallest cost dominating the committee is not specific to the exponential distribution. If the X_i are i.i.d. with *non-decreasing* residual expectation on its support, and if $c_1 < \min\{c_2, \ldots, c_m\}$, then there is an equilibrium with Player 1 using an optimal strategy as if the other players were not there, and the other players accepting any candidate agreeable to Player 1.

In the exponential case with $c_1 < \min\{c_2, \ldots, c_m\}$, this equilibrium is the unique perfect equilibrium. For other distributions with nondecreasing residual expectation, this equilibrium may not be unique. Take for example the inverse power law with density $f(x) = \alpha x^{-(\alpha+1)}$ on $(1, \infty)$ with $\alpha = 3$ (so that $EX = 3/2$ and $E(X^2) < \infty$). Suppose $m = 2$, $c_1 = 1/12$, and

$c_2 = 1/8$. The equilibrium where Player 1 dominates has equilibrium payoff $(\sqrt{6}, -\frac{3}{4}(2-\sqrt{6})) = (2.4495, -.3471)$. There is another equilibrium where Player 2 dominates that has equilibrium payoff $(5/6, 2)$.

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