Choosing Weapons for the Kinnaird Truel.

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Summary. In [5], Xiaopeng Xu has pointed out that my paper, [1], contains an error in assuming that in the Kinnaird truel [2], it is always optimal for the player with the lowest accuracy to shoot into the air, i.e. pass, when both his opponents are still alive. This paper presents a correction to Section 3 of [1], by finding the subgame perfect equilibrium for the problem of choosing weapons the Kinnaird truel without that assumption. Section 3 of [1] may be considered as a solution to the problem when the player with the lowest accuracy is required to fire into the air until one of the other contestants is hit.

1. Introduction. Three players, call them I, II and III, choose distinct probability accuracies, x, y and z respectively, in that order. Let a, b and c denote the smallest, middle and largest of these accuracies, $a < b < c \leq 1$. Let A, B and C denote the players with accuracy a, b and c respectively. Then the players play a truel in the circular order $ABCABCA\ldots$ with A starting. We only treat this protocol where the smallest accuracy shoots first, next smallest accuracy shoots second and largest accuracy shoots third. When one of the payers is eliminated, the remaining players continue in the same order until there is only one player left.

We would allow all the players to pass their turn (fire into the air), but there is a problem: it may be advantageous for each of the players to pass. For example, if a, b and c are all close to one, the first player to kill an opponent will in turn be killed in the next round with high probability. The practice of firing into the air was expressly banned by the Code Duello of 1777, [4]. So to avoid this unwanted outcome in the Kinnaird truel, we shall require at least one player, say the player with the largest accuracy, Player C, to fire at an opponent. Tom Lehrer, in Mosteller [3] points out that the code for the truel is not well developed yet, we feel we may take this liberty in this problem. Since C must fire at an opponent, it will be at B, the opponent with the next largest accuracy as long as he is alive. Therefore, Player B will not pass either, and will fire at C. We still allow Player A to pass.

2. Success Probabilities. We first find the success probabilities, $P(A \text{ wins})$, $P(B \text{ wins})$ and $P(C \text{ wins})$, in the two cases: 1. Player A is required to shoot at an opponent, which he will choose to be opponent C, and 2. Player A is required to pass.

Case 1. First consider Case 1 where, until one of the players is eliminated, Player A shoots at C and players B and C shoot at each other. With 3 players in the game, the probabilities of the first kill are

$$\begin{aligned} P_3(A \text{ kills } C) &= a/d_1 \\ P_3(B \text{ kills } C) &= (1-a)b/d_1 \\ P_3(C \text{ kills } B) &= (1-a)(1-b)c/d_1 \end{aligned}$$

where

$$d_1 = 1-(1-a)(1-b)(1-c) = a+(1-a)b+(1-a)(1-b)c.$$

Let the win probabilities be denoted by
$VA(a,b,c) = P(A \text{ wins})$, $VB(a,b,c) = P(B \text{ wins})$ and $VC(a,b,c) = P(C \text{ wins})$.

$$\begin{aligned}
VA &= P_3(A \text{ kills } C)P_2(A \text{ kills } B|B \text{ starts}) + P_3(B \text{ kills } C)P_2(A \text{ kills } B|A \text{ starts}) \\
&\quad + P_3(C \text{ kills } B)P_2(A \text{ kills } C|A \text{ starts}) \\
VB &= P_3(A \text{ kills } C)P_2(B \text{ kills } A|B \text{ starts}) + P_3(B \text{ kills } C)P_2(B \text{ kills } A|A \text{ starts}) \\
VC &= P_3(C \text{ kills } B)P_2(C \text{ kills } A|A \text{ starts})
\end{aligned}$$

where

$$\begin{aligned}
P_2(A \text{ kills } B|A \text{ starts}) &= a/d_2 \\
P_2(A \text{ kills } B|B \text{ starts}) &= (1-b)a/d_2 \\
P_2(B \text{ kills } A|A \text{ starts}) &= (1-a)b/d_2 \\
P_2(B \text{ kills } A|B \text{ starts}) &= b/d_2 \\
P_2(A \text{ kills } C|A \text{ starts}) &= a/d_3 \\
P_2(C \text{ kills } A|A \text{ starts}) &= (1-a)c/d_3
\end{aligned}$$

with

$$\begin{aligned}
d_2 &= 1-(1-a)(1-b) = a+b-ab \\
d_3 &= 1-(1-a)(1-c) = a+c-ac
\end{aligned}$$

Therefore,

$$\begin{aligned}
VA &= (a/d_1)(a(1-b)/d_2) + ((1-a)b/d_1)(a/d_2) + ((1-a)(1-b)c/d_1)((a/d_3) \\
VB &= (a/d_1)(b/d_2) + ((1-a)b/d_1)((1-a)b/d_2) \\
VC &= ((1-a)(1-b)c/d_1)((1-a)c/d_3)
\end{aligned}$$

or, simplified,

$$\begin{aligned}
VA &= a(a(1-b)+(1-a)b)/(d_1d_2) + a(1-a)(1-b)c/(d_1d_3) \\
VB &= [ab+(1-a)^2b^2]/(d_1d_2) \\
VC &= (1-a)^2(1-b)c^2/(d_1d_3).
\end{aligned} \tag{1}$$

Case 2. Now consider the where A is required to pass when it is his turn as long as all three players are alive. Then, the first-kill probabilities are

$$\begin{aligned}
P_3(B \text{ kills } C) &= b/d_4 \\
P_3(C \text{ kills } B) &= (1-b)c/d_4
\end{aligned}$$

where

$$d_4 = 1-(1-b)(1-c) = b+c-bc.$$

Here, let the win probabilities be denoted by $WA(a,b,c) = P(A \text{ wins})$, $WB(a,b,c) = P(B \text{ wins})$ and $WC(a,b,c) = P(C \text{ wins})$.

$$
\begin{aligned}
WA &= a(1-b)c/(d_3 d_4) + ab/(d_2 d_4) \\
WB &= (1-a)b^2/(d_2 d_4) \\
WC &= (1-a)(1-b)c^2/(d_3 d_4).
\end{aligned} \tag{2}
$$

Allowing Player A to fire at an opponent makes the problem more complex. In addition, the problem is complicated by another feature. One can no longer assume that Player III, facing choices of a and b ($a < b$) by his opponents, will choose just one of the three obvious choices, 1: just below a, 2: just below b, and 3: equal to 1. Indeed, it may be that if III chooses accuracy 1, Player A will shoot at him, while if he goes lower, A will pass, increasing the probability that III will win. As an example, if the choices of I and II are $a = .2$ and $b = .3$ in some order, the probabiity that III wins by choosing $c = 1$ is $VC(.2, .3, 1.) = .448$, while if III chooses $c = .778$, III's win probability is $WC(.2, .3, .778) = .488$.

For these reasons, one needs more complex computer calculations to find the equilibrium strategies. It will no longer be possible to give easily checked remarks to support the validity of the resulting equilibrium strategies. Therefore, the solution given below may be checked only by going through the process of writing a program to do the computations. Nevertheless, here are the results of a program written for this purpose.

3. Equilibrium Outcome. Let a^* and b^* denote the solution to the equations

$$
WA(b,b,1) = WB(b,1,1) \qquad \text{and} \qquad WA(a,b,b) = WC(b,b,1). \tag{3}
$$

for $0 \le a < b < 1$. The first equation of (3) expresses the indifference of III, facing choices of b and 1 by I and II, between going just below b to going just below 1, knowing that the player with the lowest accuracy will shoot into the air. The second equation expresses the indifference of II, facing a choice of b by I, between going to a and going to 1, knowing that the player with the lowest accuracy will shoot into the air.

Simplification of (3) leads to

$$
b^3 - 4b^2 + 6b - 2 = 0 \qquad \text{and} \qquad a = (1-b)^3
$$

The unique solution, subject to $0 \le a < b < 1$, is

$$
a^* = .160713\cdots \qquad \text{and} \qquad b^* = .456311\cdots.
$$

The equilibrium outcome is as follows. Player I chooses x slightly larger than b^* ($x = .456312$ suffices to six decimals); then II chooses $y = a^*$ and III chooses z slightly less than x. Player II plays the truel by firing into the air until one of the other players gets hit. The vector of winning probabilities for (I, II, III) is

$$
\big(WC(a^*,b^*,b^*), WA(a^*,b^*,b^*), WB(a^*,b^*,b^*)\big) = (.24809, .29560, .45631)
$$

One can show that Player III's payoff, $WB(a^*, b^*, b^*)$, is exactly b^*. I can see no a priori reason why this should be true.

4. Remarks. Although a full analysis showing that the above strategies form the subgame perfect (backward induction) equilibrium is complex to describe, we can present some general observations. First, one can show that for every choice of x by Player I, the strategy given by backward induction will lead the player that chooses the smallest probability to shoot into the air. In fact for $x > .22\cdots$, the strategy is exactly the same as in the case where the player that chooses the smallest probability is required to shoot into the air. Yet the overall optimal strategy without this requirement as given above is different even though the player with the smallest probability shoots into the air.

The subgame perfect equilibrium found in [2], where shooting in the air is required for the player with the smallest probability, gives the first and second players higher success probabilities. Let x^* and y^* be the solution of the equations

$$WA(x, x, y) = WB(x, y, y) = WB(x, y, 1),$$

namely, $x^* = .2026396$ and $y^* = .4688047$. In the equilibrium outcome given in [2], it was found that Player I should choose x slightly less than x^*, Player II should choose y slightly less than y^*, and Player III should choose $z = 1$. The vector of winning probabilities for the players is

$$(WA(x^*, y^*, 1), WB(x^*, y^*, 1), WC(x^*, y^*, 1)) = (.2724, .3040, .4236).$$

Why shouldn't I and II try for this result since it is better for them both. The answer is that If I chooses x slightly less than x^* and II chooses y slightly less than y^*, then III would choose z slightly less than x and shoot at II rather than in the air. Then payoff would be close to

$$(VB(x^*, x^*, y^*), VC(x^*, x^*, y^*), VA(x^*, x^*, y^*)) = (.2785, .2918, .4297)$$

which is better for I and III. But this is worse for II. The best II can do is to choose y close to $y^{**} = .3115$, say. III's best reply would be to choose z slightly less than x and then shoot into the air. This would give payoff

$$(WB(x^*, x^*, y^{**}), WC(x^*, x^*, y^{**}), WA(x^*, x^*, y^{**})) = (.1993, .3035, .4972)$$

which is very poor for I. So Player I wouldn't choose x close to .2026.

References

[1] Thomas Ferguson (2002), “Choice of Weapons for the Truel”, e-paper at http://www.math.ucla.edu/ tom/papers/unpublished/Truel.pdf.

[2] C. Kinnaird (1946) *Encyclopedia of Puzzles and Pastimes*, Citadel, Secaucus NJ, p. 246.

[3] Frederick Mosteller (1965) *Fifty Challenging Problems in Probability*, Addison-Wesley.

[4] Wikipedia Article (2012) “Duel”, at http://en.wikipedia.org/wiki/Duel

[5] Xiaopeng Xu (2012) “Game of the truel”, *Synthese* **185**, 19-25.