Choice of Weapons for the Truel

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1. Truels.

The truel is the extension of the duel from two contestants to three. There are many ways to make such an extension, several of which are discussed in the paper of Kilgore and Brams [1]. It is customary, since the problem was originally presented by Kinnaird [2] in 1946, to speak of the contestants firing at each others balloons rather than at each other. Three contestants, A, B, and C, fire in some order at each other's balloons with pistols having fixed probabilities, a, b and c, of hitting their target. The accuracies, a, b and c, are known to all players. Mosteller [4] called this problem the Three-Cornered Duel.

If simultaneous firing is allowed, it may happen at the end of a game that all balloons have been hit. We restrict attention to problems in which the players shoot in some sequential order, so that at most one pistol is fired at a time. The player, whose turn it is to shoot, may choose one of the other two balloons to fire at, or may fire at the ground (thus effectively passing). A player whose balloon is hit is out of the game, and play continues between the two remaining contestants until there is only one contestant left. That player is declared the winner and the other two players are losers. Each player wants to maximize the probability of winning.

In the Kinnaird truel, the contestants fire in the fixed sequential order, ABCABCA. . . , until only one player remains. The assignment of the players to the positions in the order is done at random with all six assignments equally likely. A restatement and solution of the problem by Larson and Moser [3] shows some surprising features. In particular, the player with the lowest accuracy always fires into the ground until there are just two players left. The other two players play a duel until one of them is put out of the game. The survivor then plays a duel against the weakest opponent with the weakest player going first. Thus it can easily happen that the weakest player has the advantage. For example, if $a = .8$, $b = .6$ and $c = .4$, with A going first, B second and C third, the respective probabilities of win, taking into account that A and B are equally likely to start in the duel between them, the respective probabilities of win are .3320, .1854, and .4826.

A somewhat smoother version of the problem was proposed by Shubik in 1954 (see [5]). In this version, at each stage the player who is allowed to shoot is determined at random (equally likely among the remaining players). Now it is to the weakest player's advantage to shoot at the strongest player's balloon, rather than into the ground. Nevertheless, the weakest player may still have the advantage. Taking the same example of accuracies $a = .8$, $b = .6$ and $c = .4$, Shubik shows that the respective probabilities of win are .296, .333 and .370. A surprising feature here is that the most accurate shooter has the smallest probability of winning. This occurs because both the other players will fire at his balloon rather than at each other's.

2. Choosing Weapons for the Shubik Truel

Before each shot in the Shubik truel, one of the players with an unbroken balloon is chosen at random equally likely to be the player who is to shoot next. The player with the surviving balloon is the winner.

Let a, b, and c denote respectively the accuracies of A, B, and C, and suppose that $a > b > c$. It seems clear that A should shoot at B's balloon since if he would prefer to face C rather than B in the two-way match. Similarly, both B and C would shoot at A since they would rather face each other than A in the two-way match. Let us assume that this is true. (Certain rules disallowing coalitions, and assumptions concerning the rationality of the players allow us to make this assumption.) Shubik gives the win probabilities as

$$\begin{aligned} P(\text{A wins}) &= a^2/(a+b+c)(a+c) \\ P(\text{B wins}) &= b/(a+b+c) \\ P(\text{C wins}) &= c(2a+b)/(a+b+c)(a+c). \end{aligned} \tag{1}$$

Shubik points out that if $a = 0.8$, $b = 0.6$, and $c = 0.4$, then the win probabilities are P(A wins) = 0.296, P(B wins) = 0.333, and P(C wins) = 0.370. The most skillful is at a disadvantage, and the least skillful has the advantage!

Suppose the players get to choose their accuracy probabilities, A chooses first, then B chooses knowing A's accuracy, and finally C chooses knowing A's accuracy and B's accuracy. Then they play Shubik's game. What accuracies should they choose, and who has the advantage?

It is assumed that a player is not allowed to choose an accuracy exactly equal to an already chosen accuracy. Furthermore, when there are still three players left, each player will shoot at the opponent with the higher accuracy.

Equilibrium Outcome. A chooses $x = .5$, then B chooses $y = 1$ and C chooses $z = .5 - \epsilon$. The payoffs are approximately 1/4, 1/3 and 5/12 respectively.

Discussion. First note that when the third player chooses his accuracy, he will choose either slightly below the lowest of a and b, or slightly below the highest of a and b, or 1. Rationality requires that he choose c to be the one of these three that gives him the highest probability of winning.

Note also that the win probabilities, (1), depend only on the ratios, b/a and c/a, say. In particular, if A chooses x and B chooses $y = x - \epsilon$, and C chooses $z = x - 2\epsilon$, then for any a, the winning probabilities are respectively, slightly greater than 1/6, slightly less than 1/3, and slightly less than 1/2.

In fact, this is what happens if A chooses $x > 1/2$. But if A chooses $x = 1/2$, then B can get 1/3 by choosing $y = 1$, forcing C to choose $z = .5 - \epsilon$.

Let γ be the root of $x^3 + 2x^2 + x - 1$ in the interval $(0, 1)$ ($\gamma = .46557123\ldots$).

If A chooses $\gamma < x < .5$, then B still chooses $y = 1$ and C chooses $z = x - \epsilon$, with B taking some of the win probability from A and C.

If A chooses $.2897 < x < \gamma$, then B chooses $y = x/\gamma$, a little less so that C will still choose $z = x - \epsilon$, keeping the win probabilities constant at $(.2410859, .3533286, .4055855)$.

If A chooses $.2884 < x < .2896$, then B chooses y to make C indifferent between $z = 1$ and $z = x - \epsilon$.

If A chooses $x < .2883$, then B chooses $y = 1$ and C chooses $z = 1 - \epsilon$, giving A a low win probability.

3. Choosing Weapons for the Kinnaird Truel

As in the Shubik truel, there are three contestants in the Kinnaird truel, but they fire in sequential order rather than random order. One of the six possible initial segments of the sequential order is chosen at random, with probability 1/6 each.

No matter which order is chosen, the two players with the two best accuracies will choose to fire at each other until one of them is eliminated. Moreover, the player with the lowest accuracy will choose not to fire at either opponent until one on them has been eliminated.

Suppose two players with accuracies x and y fight a sequential duel with the player whose accuracy is x starting. Let $f(x, y)$ denote the probability that the starting player wins. He has probability x of winning immediately, an even if he loses (probability $(1-x)$), he can still win with probability $f(x, y)$ provided his opponent misses (probability $(1-y)$). Hence, $f(x, y) = x + (1 - x)(1 - y)f(x, y)$, so that

$$f(x, y) = \frac{x}{1 - (1 - x)(1 - y)} = \frac{x}{x + y - xy}. \tag{2}$$

Now suppose players A, B, and C with respective accuracies, a, b and c fight a sequential truel with A starting, followed by B and then C. We may suppose without loss of generality that $c < a$ and $c < b$, because the player with the lowest accuracy shoots into the ground anyway and so may as well be placed last. Then

$$\begin{aligned} P_A(a, b, c) &= f(a, b)(1 - f(c, a)) = \frac{a^2(1 - c)}{(a + b - ab)(a + c - ac)} \\ P_B(a, b, c) &= (1 - f(a, b))(1 - f(c, b)) = \frac{b^2(1 - a)(1 - c)}{(a + b - ab)(b + c - bc)} \\ P_C(a, b, c) &= f(a, b)f(c, a) + (1 - f(a, b))f(c, b) \\ &= \frac{c}{a + b - ab} \cdot \frac{a(b + c - bc) + b(1 - a)(a + c - ac)}{(a + c - ac)(b + c - bc)}. \end{aligned} \tag{3}$$

The problem is complicated by the fact that the optimal choices of accuracies depend on the order of firing. To simplify, we consider the case where the order of firing order is chosen to be inversely related to the accuracies; that is the smallest accuracy shoots first, the next smallest second, and the largest last. This only affects the winning probabilities of the players with the two greatest accuracies. This is equivalent to letting the player with the middle accuracy go first, the largest accuracy go second and the smallest accuracy go third.

Player I choose his accuracy first, then Player II and finally Player III. Then the player with the middle accuracy becomes A, the player with the largest accuracy becomes B, and the other player is C.

Equilibrium Outcome. Let x^* and y^* denote the solution to the equations

$$P_C(y,x,y) = P_A(x,x,y) = P_B(x,1,y). \tag{4}$$

The solution is unique, with $x^* \approx .4688047$ and $y^* \approx .2026396$. Player I chooses a slightly less than y^*; then Player II chooses b slightly less than x^*; and finally Player III chooses $c = 1$. The vector of winning probabilities for the players is

$$(P_C(b,1,a), P_A(b,1,a), P_B(b,1,a)) \approx (.272386, .3040361, .4235253).$$

Discussion. Suppose that the accuracy choices of Players I and II are x and y in some order, where $x > y$. Then Player III will choose one of y^-, x^-, and 1. His payoff for these three choices are $P_C(y,x,y)$, $P_A(x,x,y)$ and $P_B(x,1,y)$, respectively. Player III will be indifferent among these three choices if x and y satisfy the equations (4).

Since $P_B(x,1,y) = (1-x)(1-y)$, and $P_A(x,x,y) = x(1-y)/((2-x)(x+y-xy))$, we have $P_B(x,1,y) < P_A(x,x,y)$ iff $(1-x)(2-x)(x+y-xy) < x$, or equivalently, solving for y, iff

$$y < \frac{x(-1+3x-x^2)}{(1-x)^2(2-x)}.$$

The right side is negative for $0 < x < (3-\sqrt{5})/2 \approx .382$, and greater than 1 for $.586 \approx 2-\sqrt{2} < x < 1$.

For a choice of Player I of $a >\approx .46$, Player II goes low (about $(4/9)a$) so that Player III goes just below a.

For a choice of Player I of a such that $\approx .21 < a <\approx .46$, Player II goes high (about $(9/4)a$) so that Player III goes just below a.

For a choice of Player I of $a <\approx .2$, Player II goes medium (about $(3/2)a$) so that Player III goes to 1.

References

[1] D. M. Kilgour and S. J. Brams (1997) "The Truel", *Math. Mag.* **70**, 315-326.

[2] C. Kinnaird (1946) *Encyclopedia of Puzzles and Pastimes*, Citadel, Secaucus NJ, p. 246.

[3] H. D. Larson (1948) "A dart game", *Amer. Math. Mo.* **55**, p. 248 (solution by L. Moser 640-641).

[4] Frederick Mosteller (1965) *Fifty Challenging Problems in Probability*, Addison-Wesley.

[5] Martin Shubik (1984) *Game Theory in the Social Sciences*, MIT Press, p. 22.