GAMES WITH FINITE RESOURCES

Games with finite resources are two-person zero-sum multistage games defined by Gale (1957) to have the following structure.

Player I's resource set is $A = \{1, 2, \ldots, N\}$.
Player II's resource set is $B = \{1, 2, \ldots, N\}$.
Associated with these resources is an $N \times N$ payoff matrix $\boldsymbol{M} = (M(i,j))$.

The game is played in N stages and each player is allowed to use each resource once and only once during these N stages.

At stage 1, the players simultaneously choose $a_1 \in A$ and $b_1 \in B$ and there is an immediate payoff of $M(a_1, b_1)$.

$\vdots$

At stage k, the players simultaneously choose $a_k \in A - \{a_1, \ldots, a_{k-1}\}$ and $b_k \in B - \{b_1, \ldots, b_{k-1}\}$ and there is a payoff of $M(a_k, b_k)$. It is assumed that the players know which resources have been used up to stage k.

$\vdots$

At stage N, the players use their remaining resources a_N and b_N and the payoff is $M(a_N, b_N)$. The total payoff is thus $\sum_1^N M(a_i, b_i)$.

Example 1. *Baby Goofspiel.* Player I is given the 13 hearts and Player II the 13 diamonds of an ordinary deck of cards. They simultaneously play a card from their hands and the higher card (Ace is low) wins the value of the lower card (Ace counts 1, Jack counts 11, Queen 12 and King 13). Play continues until all cards have been played (N=13 rounds). The matrix is

$$\boldsymbol{M} = \begin{array}{c} \\ \text{A} \\ 2 \\ 3 \\ \\ \text{Q} \\ \text{K} \end{array} \begin{array}{c} \begin{array}{cccccc} \text{A} & 2 & 3 & & \text{Q} & \text{K} \end{array} \\ \begin{pmatrix} 0 & -1 & -1 & \ldots & -1 & -1 \\ 1 & 0 & -2 & \ldots & -2 & -2 \\ 1 & 2 & 0 & & -3 & -3 \\ \vdots & & & \ddots & & \vdots \\ 1 & 2 & 3 & \ldots & 0 & -12 \\ 1 & 2 & 3 & \ldots & 12 & 0 \end{pmatrix} \end{array}$$

This game of finite resources is symmetric, so the value is 0. After the first move, the game may no longer be symmetric and it seems as if the optimal strategies may be quite complex.

Example 2. *Paper-Scissors-Rock.* Player I has m_1 pieces of paper, m_2 scissors, and m_3 rocks, while Player II has n_1 pieces of paper, n_2 scissors, and n_3 rocks, where $m_1 + m_2 + m_3 = n_1 + n_2 + n_3 = N$. Play is in N stages with one resource of each player used up at each stage. This game also looks quite complicated. However, Gale's remarkable result for arbitrary games of finite resources is the following.

Theorem 1. *(Gale (1957)) The value of the Game With Finite Resources is*

$$V = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} M(i, j).$$

An optimal strategy for Player I is to choose a permutation $(a_1, \ldots, a_N)$ *of* $\{1, 2, \ldots, N\}$ *at random with probability* $1/N!$ *each, and to use* a_k *at stage* k*. Similarly for Player II.*

A remarkable feature of this result is that the players' optimal strategies do not use any of the information acquired along the way. Indeed, we could alter the rules of the game so that Player II is not told the pure strategy choices of Player I as the game progresses. The lack of information does not hurt him. In addition, this strategy does not depend on the payoff matrix, $\boldsymbol{M}$. It could be that the entries to the matrix are chosen at random according to some distribution known to the players, but that only Player I is informed of the matrix that was chosen. The seeming advantage Player I gets from this information is of no use to him.

Proof. After the first stage, the players face another game of finite resources. So we use induction on N. The Theorem is trivially true for $N = 1$. Suppose it is true for $N - 1$. Then after the first move in which Player I uses a and Player II uses b, they face a game whose value is

$$V(a, b) = \frac{1}{N-1} \sum_{i \neq a} \sum_{j \neq b} M(i, j).$$

Suppose as a first move Player II uses the strategy of choosing $b = 1, 2, \ldots, N$ with equal probability $1/N$ each. The payoff to Player I if he uses a is

$$\begin{aligned}
\frac{1}{N} \sum_{b=1}^{N} \big[M(a, b) + V(a, b)\big] &= \frac{1}{N} \sum_{b=1}^{N} \Big[M(a, b) + \frac{1}{N-1} \sum_{i \neq a} \sum_{j \neq b} M(i, j)\Big] \\
&= \frac{1}{N} \Big[\sum_{b=1}^{N} M(a, b) + \frac{1}{N-1} \sum_{i \neq a} \sum_{b=1}^{N} \sum_{j \neq b} M(i, j)\Big] \\
&= \frac{1}{N} \Big[\sum_{b=1}^{N} M(a, b) + \frac{1}{N-1} \sum_{i \neq a} \sum_{j=1}^{N} \sum_{b \neq j} M(i, j)\Big] \\
&= \frac{1}{N} \Big[\sum_{b=1}^{N} M(a, b) + \frac{1}{N-1} \sum_{i \neq a} (N-1) \sum_{j=1}^{N} M(i, j)\Big] \\
&= \frac{1}{N} \sum_{b=1}^{N} \sum_{i=1}^{N} M(i, b) = V
\end{aligned}$$

Thus, Player II is guaranteed the value of the game no matter what Player I does on his first move. Similarly, Player I can also guarantee the value of the game. ■

Exercise. Find the value of the game, paper-scissors-rock, of Example 2.

References

Thomas S. Ferguson and Costis Melolidakis (2000) "Games with finite resources", *Inter. J. Game Theory*, **45**, 289-303.

David Gale (1957) "Information in games with finite resources", *Ann. Math. Studies* **39**, 141-145.